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II. *On a Cassegrain Reflector with Corrected Field.*

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THE great advantage enjoyed by the reflecting telescope is its equal treatment of rays of all colours, and the geometrical defects or aberrations of its field are less than those of many of the older refractors. The most serious of these defects is coma, owing to which different zones of the objective do not place the light which they receive from the same object-point symmetrically around any common centre in the image area, but arrange it in a radial fan or flare, the light from the outer zones being most diffused; besides spoiling the image this tends to neutralize, for any except narrow fields, the value of extended aperture in the objective as a light-collector. In the refractor this can be and is now always met by adjusting the curves of the two lenses, for when achromatism, as far as possible, and spherical aberration are allowed for, there still remains one unused datum; in old forms this was often used to make the inner curves contact curves that might be cemented together if it was convenient to do so, but it is properly employed to extinguish coma. But with the reflector the case is different. In the Newtonian form there is only one available surface, and when this is made a paraboloid to cure spherical aberration, nothing is left to adjust. In the Gregorian or Cassegrain forms there are two curved surfaces and, theoretically, these would offer means to correct two faults. An illuminating study of the possibilities of a system of two mirrors has been made by SCHWARZSCHILD in his 'Untersuchungen zur Geometrischen Optik';* I shall deal with its outcome below. Its general tenor is comprehensive and exploratory rather than detailed, and it remains doubtful whether any of the forms which he indicates for the reflector, at the point at which his research stops, could actually be made successfully upon a scale that would show their advantages. My own purpose in the present paper is essentially a practical one. I have in mind throughout a telescope of large aperture and considerable focal length, and seek to devise a correction for the faults of its field which shall leave its achromatism unimpaired, which can really be made and which shall effect its purpose without employing any curves and angles outside those that are already known to work well. It has been

* 'K. Gesell. d. Wissenschaften zu Göttingen, Abhandl. Math.-Phys. Classe,' Neue Folge, Bd. IV., 1905.

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said that "an object-glass cannot be made on paper," but the possibilities of new and somewhat complicated constructions must in all cases first be demonstrated on paper, since practice can never conveniently vary more than a single factor at a time. Study is directed to the Cassegrain because of the great advantage which this design possesses in shortening the tube of the instrument for given focal length, and in placing the observer at the lower, in place of at the upper, end of it.

The best introduction to the subsequent work will be in the form of a few remarks upon SCHWARZSCHILD'S results. These are not meant as a complete criticism or estimation of it but are merely such as arise naturally in relation to the points with which I deal afterwards. The traditional form of Cassegrain telescope consists of a great concave mirror faced by a small convex one, which is placed between the great mirror and its principal focus, and throws the image out through a hole cut centrally in the great mirror. The small mirror increases the effective focal length in the ratio of its distances respectively from the final principal focus and from the principal focus of the great mirror. This ratio for example is 5·4 in the great Melbourne telescope, $3\frac{1}{5}$ to $4\frac{4}{5}$ in the Mount Wilson 60-inch when used as a Cassegrain, and it can hardly fall much below $2\frac{1}{2}$ unless the small mirror is to cut off a disproportionate amount of the area of the great mirror. The Cassegrain is, therefore, generally speaking, a long focus instrument. From all these features SCHWARZSCHILD'S forms differ widely, except that they place the small mirror between the great mirror and its principal focus. His small mirror is concave in place of convex, and shortens the effective focal length, bringing the beam to a focus between itself and the great mirror. The effect of this change in design is to render possible a flat field. Spherical aberration and coma are removed from the image by modifying the spherical figures of the two mirrors into definite hyperboloidal and ellipsoidal forms. To confine reference to the case which he considers generally the best (*loc cit.*, II., § 11), the necessary deformations are given respectively by $b_1 = -13{\cdot}5$, $b_2 = +1{\cdot}97$, where $b = -1$ would deform a sphere into a paraboloid. The image-surface for this case would be very nearly flat, and the images of points would be very nearly circles, which reached a diameter of ·8 seconds at an angular distance of about 1 degree from the centre of the field. This may seem somewhat large but it is a quantity proportional to the aperture-ratio, which in this case is large also, namely $1:3{\cdot}5$. The result is in brief a very rapid instrument of short focus and of field about comparable to that of a good long-focus refractor. The chief objection to it is found in the curves that it requires. Until some one turns such curves out, it must remain problematic whether it is feasible at all to make the construction a practical success.

A feature of SCHWARZSCHILD'S analysis is the use of a concave small mirror. This is not necessary to destroy coma, which may equally be removed in the Cassegrain form by deformations of the mirrors, and those indeed of less pronounced degree than SCHWARZSCHILD finds necessary. But as will be shown below there then remains a somewhat severe and irremovable curvature of the field.

The general conclusion which I draw from SCHWARZSCHILD'S investigation is that modification of the two mirrors is in itself not enough to give a practical solution of the problem. We have to deal with spherical aberration, coma, curvature of the field, and astigmatism. Distortion may be set aside, because in itself it does not vitiate the image of a point, and errors which it introduces into relative distances may be computed and allowed for. We have at our disposal the figures of the two mirrors and their separation and curvatures. The last are so locked up with the kind of telescope which we wish to produce that they are hardly available for adjustment—if we want a short-focus instrument we have to take SCHWARZSCHILD'S choice, and for a long-focus one the Cassegrain form. It turns out that the former of these may have a flat field and the latter must have a curved field and we have to rest content with that. And with respect to the figures of the mirrors it is not within our control to say whether they shall offer themselves in our equations in a favourable form for removing undesired terms; it appears from the research that they appear somewhat unfavourably entailing the use of surfaces decidedly far from the sphere. It is my object to obtain a workable solution and not merely a theoretical one, and therefore I have recourse to a more complicated system, by passing the beam through a definite set of lenses, the curvatures of which are more or less completely at our disposal. It might, at first sight, appear that this would impair the achromatism of the reflector, but if a system of not less than three separated lenses be made of the *same glass*, the two conditions for achromatism at a given plane may be completely satisfied, equally for all colours. With such a system we can produce deviation in a beam, but more emphatically we can produce aberrations. The details at which I arrive are given on p. 66, and need not be repeated here, but generally the plan is to replace the convex mirror by a weak convexo-concave lens silvered at the back, and about two-thirds of the way between this and the surface of the great mirror to place a system which I call the Corrector, being a pair of lenses of nearly equal but opposite focal lengths, of which the first is double concave with the lesser curvature first, and the latter nearly plano-convex.

Choosing the curvatures properly a telescope is thus produced which gives, strictly in the focal plane, an image free from chromatic faults, except for minute chromatic residues of aberration, from spherical aberration and from coma, and in which points of the object are represented in the image by spots strictly circular which reach a diameter of 2·2 seconds at a distance of 1 degree from the centre of the field. The greatest angle of incidence upon any of the surfaces is 11 degrees, or not more than about two-thirds of what is customary upon the anterior surface of the flint lens of the object glass of a refractor; all the surfaces are spherical except that of the great mirror which is intermediate between the sphere and paraboloid, and I cannot see that anywhere any serious constructional difficulty is introduced. The effective aperture-ratio is 1 : 14·05, or, say, about 1 : 15, allowing that 12 per cent. more light will be lost in this construction than in other possible ones.

The methods which I employ are those of a memoir recently published.* SCHWARZSCHILD used the Characteristic Function. Our methods thus differ, but since aberrations of the third or any other order are the same things, no matter how they are obtained, where we occasionally touch the same matter the differences are at most those of notation, and occasionally these are slight ones. I have not attempted to remove them because it seems to me that an investigation is easiest to read if expressed in notation that grows naturally out of its own processes. I shall therefore adhere strictly to the notation of my Memoir, amplifying its results as occasion requires.

We may take for reference the following specifications of the faults of an optical field at its principal focus in terms of the coefficients $\delta_1 G$, &c. :—

a = semi-aperture.

f' = effective focal length.

β = tangent of inclination of original ray to axis.

Position of least circle of spherical aberration . . .	$\delta f' = +\frac{3}{8} f' a^2 \delta_1 G$.
Angular radius of this circle	$25783'' \times \dfrac{a}{f'} \times a^2 \delta_1 G$.
Comatic radius	$103133'' \times \dfrac{a}{f'} \times a\beta\delta_2 G$.
Secondary focal line after principal focus	$\frac{1}{2} f' \beta^2 \delta_3 G$.
Primary focal line after secondary	$f' \beta^2 \delta_2 H$.
Radius of focal circle	$103133'' \times \dfrac{a}{f'} \times \beta^2 \delta_2 H$.
Curvature of field (convex to ray if positive) . . .	$(1/f') \times (\delta_3 G + \delta_2 H)$.
Distortional displacement	$103133'' \times (1/f') \times \beta^3 \delta_3 H$.

. . . . (1)

With respect to these it may be explained that the Comatic Radius is the radius of the circle around which rays from a zone of radius a are distributed, the centre of the comatic circle being displaced from the normal image-point by an amount equal to its diameter; the "secondary" focal line is the line in the plane of the axis; the word "after" means after, in the order in which light reaches the points; the focal circle is the circle half way between the two focal lines, through which, in the absence of coma, all rays of the zone would pass; the curvature of the field refers to the field containing the focal circles of all object-points.

Now, if we secure a field for which

$$\delta_1 G = 0, \qquad \delta_2 G = 0, \qquad \delta_3 G + \delta_2 H = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* "A New Treatment of Optical Aberrations," 'Phil. Trans.,' vol. 212, pp. 149–185.

it will be free from spherical aberration and from coma, and the images of points will be circles in the plane through the principal focus, the radii of which are given by $103133'' \times (a/f') \times \beta^2 \delta_2\mathrm{H}$. If $\delta_2\mathrm{H}$, which by (2) is made equal to $-\delta_3\mathrm{G}$, is not zero, the instrument will be successful for such values of the angular radius of the field as keep this down below desired limits. These conditions give the objects which I aim at attaining. Given the general design of the instrument as regards apertures and focal lengths, it will be found that the lens which is used as a mirror, or the Reverser as I shall call it, is completely determined in its curvatures by the conditions for achromatism, and the quantities available for adjustment are the figure of the great mirror and the curvatures of the two lenses of the corrector. These are used to satisfy rigorously equations (2), and the essential difficulty of the problem is to find a case among the great number of those that are open for trial, the solution of which shall prove to be of a practical kind, not involving excessive curvatures. Once an approximate solution is obtained, to refine it only requires patience, but to arrive in the neighbourhood of a solution is a problem in which trial needs some guide. In this connection I would draw attention to the theory given below of the Thin Corrector. This is an optical system of two or more thin lenses in contact, null as far as deviation and colour are concerned, and introducing aberrations only which are available for correcting existing aberrations. Thus simplified, it is manageable algebraically, and its indications will show the possibility or otherwise of any projected arrangement.

If we denote by $\mathfrak{B}$ the curvature of the field and by $\mathfrak{P}$ PETZVAL's expression

$$\mu_{-1}\Sigma\,(1/\mu_{2r+1}-1/\mu_{2r-1})\,\mathrm{B}_{2r},$$

B_{2r} being the curvature of the surface $(2r)$, as in the Memoir, p. 162, we have

$$\delta_3\mathrm{G}+\delta_2\mathrm{H} = f'\mathfrak{B}, \qquad \delta_3\mathrm{G}-\delta_2\mathrm{H} = \mathrm{H}\mathfrak{P} = f'\mathfrak{P}$$

at the principal focus; hence $\delta_2\mathrm{H}$ which gives the amount of astigmatism is determined by

$$\delta_2\mathrm{H} = \tfrac{1}{2}(\mathfrak{B}-\mathfrak{P})f' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

a result which can also be deduced at sight from known expressions for astigmatism and curvature of field according to SEIDEL's theory. In the special case of a flat field, or $\mathfrak{B} = 0$, it becomes

$$\delta_3\mathrm{G} = -\delta_2\mathrm{H} = \tfrac{1}{2}\mathfrak{P}f' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\mathrm{A})$$

and this may be taken in place of the third of equations (2) as one of our necessary conditions. We notice that it is only possible to control the astigmatism through the value of $\mathfrak{P}$, and the value of $\mathfrak{P}$ depends only in small degree upon the distribution of curvatures between the two faces of a lens. It is a matter then of the general design of the instrument to keep $\delta_3\mathrm{G}$ down to a suitable magnitude. This presents no difficulty. I have been content to keep it small enough for my purpose. If a field of radius greater than 1 degree were desired, it could be made even smaller, but it would seem to involve the sacrifice of some other conveniences.

The values of the quantities δ_1G, &c., for the combined system are built up step by step by proceeding from surface to surface or from lens to lens by the sequence equations (17), p. 160, of the Memoir referred to above. For making these steps it is not convenient to lay down any one procedure as being the best for all cases, but two methods may be mentioned, one or other of which is frequently suitable. First we can proceed from conjugate focus to conjugate focus, the first focus being the principal focus of the first or great mirror, and each successive conjugate focus being the principal focus of the whole combination which precedes it. That is to say, at each stage we have

$$g = 0, \qquad hk = -1, \qquad h' = 0,$$

so that the equations we require to consider are

$$\begin{aligned} \delta_1 G &= g'\delta_1 g + k^3\delta_3 h', \\ \delta_2 G &= g'\delta_2 g - k\delta_2 h' + k^2 l\delta_3 h', \\ \delta_3 G &= g'\delta_3 g + k^{-1}\delta_1 h' - 2l\delta_2 h' + kl^2\delta_3 h'. \qquad (4) \end{aligned}$$

In these $g', \ldots$ refers to the new or added element, $g, \ldots$ to the combination from the beginning up to this element, and G, ... to the resulting combination including this element. We thus notice that $\delta_s g$ contributes to δ_sG simply by multiplying by g', which is the magnification of the new element between its conjugate foci under consideration. We notice, too, that so long as we confine ourselves to δ_sG, the only coefficients which it is necessary to find for each added element are $\delta_s h'$, calculated between the same conjugate foci. If the aberrations of the second element are given, referred to some other origins, they must be transferred to the conjugate foci in question by means of the equations for change of origin (22), p. 164. A case will present itself that requires a modification of this process, namely, when one of the conjugate foci belonging to an element introduced by one of the steps described is at a great distance; to meet this case we may take this element together with the next following one and combine them into one before adding them to the combination, or we may take a second completely different method as follows:—

Let O_0, O_n be the initial and final origins; O_a, $O_{a'}$ the origins to which the known aberrations of a part of the system are referred. Calling $\{g', h'; k', l'\}$ the subsequent normal system $O_{a'}$ to O_n, transfer the aberrations to origins $O_a \ldots O_n$ by use of the first part of equations (17), p. 160, viz., $\delta_1 G = g'\delta_1 g + h'\delta_1 k, \ldots$. Then calling $\{g, h; k, l\}$ the preceding normal scheme O_0 to O_a, transfer the so-found coefficients from $O_a \ldots O_n$ to $O_0 \ldots O_n$ by using the forms of the second part of the same equations. An example of this method will be found on p. 55.

We now study the formulæ for thin lenses. It will be pointed out later how to make use of these when the lenses are thick.

Thin Lenses.

The aberration coefficients for a single surface are given in the Memoir, p. 161;

$$\delta_1 g = (1-n)\,B^2, \quad \delta_2 g = 0, \quad \delta_3 g = 0, \quad \delta_1 h = (1-n)\,B, \quad \delta_2 h = 0, \quad \delta_3 h = 0,$$

$$\delta_1 k = (1-n)(-1+n-n^2+e)\,B^3, \quad \delta_2 k = -n^2(1-n)\,B^2, \quad \delta_3 k = -n(1-n^2)\,B,$$

$$\delta_1 l = (1-n)(-1+n-n^2)\,B^2, \quad \delta_2 l = -n^2(1-n)\,B, \quad \delta_3 l = -n(1-n^2), \quad . \quad . \quad (5)$$

where I have written $e = 1-\epsilon$, so that $e = 0$ for a spherical surface, and $e = 1$ for a paraboloid.

Both origins are at the surface, and

$$g = 1, \quad h = 0, \quad k = (n-1)\,B, \quad l = n, \quad \mathfrak{p} = k, \quad n = \mu_{-1}/\mu_{+1}.$$

The case of the thin lens, with origins at its surface, is derived from this by an application of equations (17), p. 160.

Write

$$k = \left(1-\frac{1}{n}\right)(B-B'), \qquad \mathfrak{p} = kn, \qquad q = \left(1+\frac{1}{n}\right)(B+B'),$$

then

$$g = 1, \qquad h = 0, \qquad k = \left(1-\frac{1}{n}\right)(B-B'), \qquad l = 1,$$

$$\begin{aligned}
\delta_1 g &= -\tfrac{1}{2}kn\,(k+q) = -\tfrac{1}{2}\mathfrak{p}\,(k+q), \qquad \delta_2 g = 0, \qquad \delta_3 g = 0,\\
\delta_1 h &= -kn = -\mathfrak{p}, \qquad \delta_2 h = 0, \qquad \delta_3 h = 0,\\
\delta_1 k &= \left\{1+\tfrac{1}{4}\frac{n\,(2-n)}{(1-n)^2}\right\}k^3+\tfrac{1}{2}nk^2q+\tfrac{1}{4}\,\frac{n^2\,(1+2n)}{(1+n)^2}\,kq^2+\left(1-\frac{1}{n}\right)(eB^3-e'B'^3),\\
\delta_2 k &= k^2-\delta_1 g,\\
\delta_3 k &= k\,(1+n) = k+\mathfrak{p},\\
\delta_1 l &= \delta_2 k-k\mathfrak{p} = k^2-k\mathfrak{p}-\delta_1 g,\\
\delta_2 l &= k,\\
\delta_3 l &= 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)
\end{aligned}$$

It may be mentioned that B, the curvature, is positive when the convex face is presented to the ray.

It seems unnecessary to give the algebra leading to these expressions in all cases. It is quite straightforward, and that for $\delta_1 k$, which is relatively long, may be taken as a model. From the Memoir, p. 160, we have, taking $\delta_1 K$ to refer to the joint effect of the two surfaces

$$\delta_1 K = k'\delta_1 g+l'\delta_1 k+\{\delta_1 k'+2k\delta_2 k'+k^2\delta_3 k'\}+k\{\delta_1 l'+2k\delta_2 l'+k^2\delta_3 l'\}.$$

It is clear that the terms in e, e' come to the values given. Leaving these aside

$$\delta_1 l' + 2k\delta_2 l' + k^2\delta_3 l' = \mathrm{B}'^{-1}\{\delta_1 k' + 2k\delta_2 k' + k^2\delta_3 k'\}$$

$$= \left(1-\frac{1}{n}\right)\left(-1+\frac{1}{n}-\frac{1}{n^2}\right)\mathrm{B}'^2 - 2(1-n)\,\mathrm{B}\,.\,(-n^2)\left(1-\frac{1}{n}\right)\mathrm{B}' + (1-n)^2\mathrm{B}^2\,.\,\frac{1}{n}\left(-1+\frac{1}{n^2}\right)$$

$$= n^{-3}[(1-n)^3(1+n)\,\mathrm{B}^2 - 2(1-n)^2\,\mathrm{BB}' + (1-n)(1-n+n^2)\,\mathrm{B}'^2].$$

This appears, multiplied by $-(1-n)\,\mathrm{B}+\mathrm{B}'$, and added to $k'\delta_1 g + l'\delta_1 k$ which is

$$-\left(1-\frac{1}{n}\right)\mathrm{B}'\,.\,(1-n)\,\mathrm{B}^2 + \frac{1}{n}(1-n)(-1+n-n^2)\,\mathrm{B}^3;$$

the whole is

$$\begin{aligned}
&\mathrm{B}^3\times && \frac{1}{n}(1-n)(-1+n-n^2) - \frac{1}{n^3}(1-n)^3(1-n^2) = -\frac{1}{n^3}(1-n)(1-2n+n^2+n^3)\\
&+\mathrm{B}^2\mathrm{B}'\times && \frac{1}{n}(1-n)^2 + \frac{2}{n^3}(1-n)^3 + \frac{1}{n^3}(1-n)^2(1-n^2) = \frac{1}{n^3}(1-n)(3-5n+2n^2)\\
&+\mathrm{BB}'^2\times && -\frac{1}{n^3}(1-n)^2(1-n+n^2) - \frac{2}{n^3}(1-n)^2 = -\frac{1}{n^3}(1-n)(3-4n+2n^2-n^3)\\
&+\mathrm{B}'^3\times && \frac{1}{n^3}(1-n)(1-n+n^2).
\end{aligned}$$

This may be written

$$\begin{aligned}
&-n^{-3}(1-n)(\mathrm{B}-\mathrm{B}')[(1-2n+n^2+n^3)\,\mathrm{B}^2 + (-2+3n-n^2+n^3)\,\mathrm{BB}' + (1-n+n^2)\,\mathrm{B}'^2]\\
&= -n^{-3}(1-n)(\mathrm{B}-\mathrm{B}')[(1-n)^2(\mathrm{B}-\mathrm{B}')^2 + n^3\mathrm{B}^2 + (-n+n^2+n^3)\,\mathrm{BB}' + n\mathrm{B}'^2]\\
&= \mathrm{K}^3 + \mathrm{KX}
\end{aligned}$$

where

$$\mathrm{X} = n\mathrm{B}^2 + (-n^{-1}+1+n)\,\mathrm{BB}' + n^{-1}\mathrm{B}'^2,$$

$$= \tfrac{1}{4}\frac{n(2-n)}{(1-n)^2}\mathrm{K}^2 + \tfrac{1}{2}n\mathrm{KQ} + \tfrac{1}{4}\frac{n^2(2n+1)}{(1+n)^2}\mathrm{Q}^2. \quad \ldots\ldots \quad (7)$$

This is the given expression if finally we write small letters for capitals.

It will be noticed that q, which contains the reference to the distribution of curvatures, apart from their effect upon focal length only presents itself in the forms in which it is introduced by $\delta_1 k$, $\delta_1 g$. It is somewhat remarkable that the same is true when we have any number of thin lenses in contact; thus, if we have a system of thin lenses in contact, giving a set of coefficients $\delta_1 g$, ..., and add a single thin lens to it for which we have $\delta_1 g'$, ..., then, noticing that

$$g = l = 1, \qquad h = 0, \qquad g' = l' = 1, \qquad h' = 0,$$

we have

$$\begin{aligned}
\delta_1 \mathrm{G} &= \delta_1 g + \delta_1 g' + k\delta_1 h' = \delta_1 g + \delta_1 g' - k\mathfrak{p}', \qquad \delta_2 \mathrm{G} = \delta_3 \mathrm{G} = 0, \\
\delta_1 \mathrm{H} &= \delta_1 h + \delta_1 h' = -\mathfrak{p} - \mathfrak{p}' = -\mathfrak{P}, \qquad \delta_2 \mathrm{H} = \delta_3 \mathrm{H} = 0, \\
\delta_1 \mathrm{K} &= \delta_1 k + k'\delta_1 g + \{\delta_1 k' + 2k\delta_2 k' + k^2 \delta_3 k'\} + k\,\{\delta_1 l' + 2k\delta_2 l' + k^2\delta_3 l'\}, \\
&= \delta_1 k + \delta_1 k' + k'\delta_1 g + 2k\,(k'^2 - \delta_1 g') + k^2\,(k' + \mathfrak{p}'), \\
&\qquad\qquad + k\,(k'^2 - k'\mathfrak{p}' - \delta_1 g') + 2k^2 k' + \mathrm{E}, \\
&= (k+k')^3 + k\chi + k'\chi' + k'\delta_1 g - 3k\delta_1 g' + k\,(k-k')\,\mathfrak{p}' + \mathrm{E},
\end{aligned}$$

where E is the sum of terms in e, e' for each of the lenses;

$$\begin{aligned}
\delta_2 \mathrm{K} &= \delta_2 k + \delta_2 k' + k\delta_3 k' + k\delta_2 l', \\
&= k^2 - \delta_1 g + k'^2 - \delta_1 g' + k\,(k' + \mathfrak{p}') + kk', \\
&= (k+k')^2 - (\delta_1 g + \delta_1 g' - k\mathfrak{p}') = \mathrm{K}^2 - \delta_1 \mathrm{G}, \\
\delta_3 \mathrm{K} &= \delta_3 k + \delta_3 k' = k + k' + \mathfrak{p} + \mathfrak{p}' = \mathrm{K} + \mathfrak{P}, \\
\delta_1 \mathrm{L} &= \delta_2 \mathrm{K} - \mathrm{K}\mathfrak{P} = \mathrm{K}^2 - \mathrm{K}\mathfrak{P} - \delta_1 \mathrm{G}, \\
\delta_2 \mathrm{L} &= \delta_3 \mathrm{K} - \mathfrak{P} = \mathrm{K}, \qquad \delta_3 \mathrm{L} = 0. \qquad \qquad (8)
\end{aligned}$$

Thus, to form the coefficients $\delta_1\mathrm{G}$, ... for any system of thin lenses in contact, we require to know only the forms for $\delta_1\mathrm{G}$ and $\delta_1\mathrm{K}$. I add the forms of these for three lenses,

$$\begin{aligned}
\delta_1 \mathrm{G} &= \delta_1 g + \delta_1 g' + \delta_1 g'' - k\,(\mathfrak{p}' + \mathfrak{p}'') - k'\mathfrak{p}'', \\
&= -\tfrac{1}{2}\,(\mathfrak{p}q + \mathfrak{p}'q' + \mathfrak{p}''q'') - \tfrac{1}{2}\mathfrak{p}k - \tfrac{1}{2}\mathfrak{p}'\,(2k + k') - \tfrac{1}{2}\mathfrak{p}''\,(2k + 2k' + k''), \\
\delta_1 \mathrm{K} &= \mathrm{K}^3 + k\chi + k'\chi' + k''\chi'' + \mathrm{E}, \\
&\qquad + (k' + k'')\,\delta_1 g + (-3k + k'')\,\delta_1 g' + (-3k - 3k')\,\delta_1 g'', \\
&\qquad + k\,(k - k' - k'')\,\mathfrak{p}' + (k + k')\,(k + k' - k'')\,\mathfrak{p}'', \\
&= \mathrm{K}^3 + k\chi + k'\chi' + k''\chi'' + \mathrm{E}, \\
&\qquad - \tfrac{1}{2}\,(k' + k'')\,\mathfrak{p}q + \tfrac{1}{2}\,(3k - k'')\,\mathfrak{p}'q' + \tfrac{3}{2}\,(k + k')\,\mathfrak{p}''q'', \\
&\qquad - \tfrac{1}{2}k\,(k' + k'')\,\mathfrak{p} + \tfrac{1}{2}(2k + k')\,(k - k'')\,\mathfrak{p}' + \tfrac{1}{2}(2k + 2k' + k'')\,(k + k')\,\mathfrak{p}''. \qquad (9)
\end{aligned}$$

From these, if necessary, the general case may be written down by analogy without much difficulty, *e.g.*, in $\delta_1\mathrm{K}$ the coefficient of $\tfrac{1}{2}\mathfrak{p}'q'$ is three times the k of the preceding system *minus* the k of the following system; but I shall not require more than three.

We may employ these equations where we require to obtain algebraically rough but reliable indications of the properties of a given actual system. Thus, consider the aberrations of any set of thin lenses in contact, at their principal focus, that is, at a distance $-\mathrm{K}^{-1}$ beyond their common surfaces. We must form $\delta_1\Gamma = \delta_1\mathrm{G} - \mathrm{K}^{-1}\delta_1\mathrm{K}$, ... where $\delta_1\mathrm{G}$, ... are the quantities just found which refer to the surfaces of the lenses as origins. Hence for example, referring to p. 30, we see that the radius of the focal circle, and the separation of the focal lines is constant in such a system, the former being equal to $103133'' \times (a/f') \times \beta^2$, and the latter to $f'\beta^2$. The curvature of the field is $2K+\mathfrak{P}$, or the radius of curvature is always about two-fifths of the focal length.

The condition for absence of coma, which is usually given as ABBE's Sine Condition, may be put

$$0 = K\delta_2\Gamma = K\delta_2 G - \delta_2 K = \delta_1 G - K^2;$$

in this the right-hand member, apart from the focal lengths, is a linear function of the quantities q.

The condition for absence of spherical aberration is

$$0 = K\delta_1\Gamma = K\delta_1 G - \delta_1 K,$$

which is a quadratic function of $q, \ldots$.

A numerical example of the use of such approximations will be given later.

It is necessary to deal with express care with the case of the mirror. It may be treated as a single surface for which $n = -1$, and then

$$\begin{array}{llllll} g = 1, & h = 0, & k = -2B, & l = -1, & \mathfrak{p} = -2B, & \\ \delta_1 g = 2B^2, & \delta_2 g = 0, & \delta_3 g = 0, & \delta_1 h = 2B, & \delta_2 h = 0, & \delta_3 h = 0, \\ \delta_1 k = -2(3-e)B^3, & \delta_2 k = -2B^2, & \delta_3 k = 0, & & & \\ \delta_1 l = -6B^2, & \delta_2 l = -2B, & \delta_3 l = 0, & & & \end{array}$$

but this leaves the positive axis after reflection opposite to the direction of the ray. It is better to reverse the direction of the axis, and this may best be done by multiplying by the scheme $\{g, h;\ k, l\} \equiv \{1, *;\ *, -1\}$, and gives the following set to represent the mirror :—

$$\begin{array}{lllll} g = 1, & h = 0, & k = 2B, & l = +1, & \mathfrak{p} = -2B, \\ \delta_1 g = 2B^2, & \delta_2 g = \delta_3 g = 0, & \delta_1 h = 2B, & \delta_2 h = \delta_3 h = 0, & \\ \delta_1 k = 2(3-e)B^3, & \delta_2 k = 2B^2, & \delta_3 k = 0, & & \\ \delta_1 l = 6B^2, & \delta_2 l = 2B, & \delta_3 l = 0, & & \end{array} \qquad (10)$$

the signs of all terms in k, l being reversed by this step, while $g, h, \mathfrak{p}$ remain unchanged. Notice that the convention for the sign of B has not been altered, so that, *e.g.*, for the concave mirror B is negative, and the new value of $k = (1-n)B$ is negative also.

If we write $\delta_1 k = k^3 + k\chi + E$, we must put $\chi = -\frac{1}{4}k^2$.

Besides the simple mirror I shall have also to deal with the system consisting of a meniscus, silvered at the back. Such a system I shall call a Reverser. For neglected thickness the coefficients follow readily from the case above (p. 35), of the juxtaposition of three thin lenses, replacing the middle lens by a mirror, and taking for the third lens the original lens with the surfaces in reversed order. This reversal of order will replace B, B′ respectively by $-B'$, $-B$. Hence $k, \mathfrak{p}$ will equal $k'', \mathfrak{p}''$ respectively, but $q+q''=0$.

Hence in the expressions (9), using ′ to denote the mirror surface

$$\delta_1 g+\delta_1 g'' = -k\mathfrak{p}, \qquad \delta_1 g' = 2B'^2 = -\tfrac{1}{2}k'\mathfrak{p}',$$

$$\delta_1 G = -k\mathfrak{p}-\tfrac{1}{2}k'\mathfrak{p}'-k(\mathfrak{p}+\mathfrak{p}')-k'\mathfrak{p} = -\tfrac{1}{2}(2k+k')(2\mathfrak{p}+\mathfrak{p}') = -\tfrac{1}{2}K\mathfrak{P}. \quad . \quad . \quad (11)$$

The same expression is true of a more complicated reverser of any number of thin lenses with the last surface silvered. Also

$$\begin{aligned}\delta_1 K = {} & K^3+(k\chi+k'\chi'')-\tfrac{1}{4}k'^3+E \\ & -\tfrac{1}{2}(k+k')\mathfrak{p}(k+q)-2k(-\tfrac{1}{2}k'\mathfrak{p}')+\tfrac{3}{2}(k+k')\mathfrak{p}(k-q) \\ & +k(-k')\mathfrak{p}'+(k+k')k'\mathfrak{p} \\ = {} & K^3+\frac{2n-n^2}{2(1-n)^2}k^3+\frac{n^2(1+2n)}{2(1+n)^2}kq^2-\tfrac{1}{4}k'^3+E \\ & -2(k+k')\mathfrak{p}q+(k+k')^2\mathfrak{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)\end{aligned}$$

To conclude this preliminary discussion of systems of thin lenses in contact I shall introduce a system which consists of two thin lenses in contact, of equal and opposite focal length and of the same glass, and therefore a null system in every respect except for aberrations. The use of such a system will be illustrated hereafter. Its simplicity is such that its aberration-coefficients reduce to very easy forms, and can therefore be handled algebraically in an experimental investigation, in order to discover what system will correct the aberrations of a proposed system; it will supply a useful approximation to a solution when any less idealised system is too complicated to manage.

From the expressions (8) we have for the Thin Corrector

$$K = k+k' = 0, \qquad \mathfrak{P} = kn+k'n = 0,$$

$$\begin{aligned}\delta_1 G = {} & -\tfrac{1}{2}k^2 n(1+q/k)-\tfrac{1}{2}k'^2 n(1+q'/k')-kk'n = -\tfrac{1}{2}k^2 n(q/k+q'/k'), \\ \delta_1 K = {} & (k+k')^3+E \\ & +k^3\left\{\frac{2n-n^2}{4(1-n)^2}+\tfrac{1}{2}n(q/k)+\frac{n^2(1+2n)}{4(1+n)^2}(q/k)^2\right\} \\ & +k'^3\left\{\frac{2n-n^2}{4(1-n)^2}+\tfrac{1}{2}n(q'/k')+\frac{n^2(1+2n)}{4(1+n)^2}(q'/k')^2\right\} \\ & -\tfrac{1}{2}k'k^2 n(1+q/k)+\tfrac{3}{2}kk'^2 n(1+q'/k')+k(k-k')k'n \\ = {} & k^3\left[n\{q/k+q'/k'\}+\frac{n^2(1+2n)}{4(1+n)^2}\{(q/k)^2-(q'/k')^2\}\right]+E, \quad . \quad . \quad . \quad . \quad (13)\end{aligned}$$

and all the rest of the coefficients run in agreement with p. 35, so that

$\delta_2 K = \delta_1 L = -\delta_1 G$ and the rest are zero. These are the values at the surface of the corrector. We notice that all are zero when $q/k + q'/k' = 0$, that is, when the curvatures of the two surfaces in contact are the same.

In order to illustrate the manner of using these, for example, let it be proposed to find the curvatures of a corrector, which when interposed at a given point of an aberrant beam shall produce assigned changes in it. Let this place be at a distance v before the beam comes to its focus. After passing through the corrector it will still come to a focus at the same place, so that applying the formulæ of the Memoir, p. 164, (22), we have for the distances from the first conjugate focus to the corrector $d = v$, which is negative, and from the corrector to the second conjugate focus $d' = -v$, and transferring from the surface of the corrector to these conjugate foci, we have

$$\delta_1 h' = 2v\delta_1\gamma - v^2\delta_1\kappa,$$
$$\delta_2 h' = 3v^2\delta_1\gamma - v^3\delta_1\kappa,$$
$$\delta_3 h' = 4v^3\delta_1\gamma - v^4\delta_1\kappa,$$

where $\delta_1\gamma$, $\delta_1\kappa$ are written for the values of $\delta_1 G$, $\delta_1 K$ given in (13).

We must now apply the formulæ (4) of p. 32. For the corrector $g' = 1$. Let the assigned changes be, say,

$$\Delta_2 = \delta_2 G - \delta_2 g, \qquad \Delta_3 = \delta_3 G - \delta_3 g,$$

so that the equations (4) of p. 32 give

$$k^{-1}\Delta_2 = -\delta_2 h' + kl\delta_3 h' = (-3 + 4klv)\, v^2\delta_1\gamma + (1 - klv)\, v^3\delta_1\kappa,$$
$$k\Delta_3 - l\Delta_2 = \delta_1 h' - kl\delta_2 h' = (+2 - 3klv)\, v\delta_1\gamma - (1 - klv)\, v^2\delta_1\kappa,$$

therefore

$$v^2\delta_1\gamma = -\Delta_2 k^{-1} - \Delta_3 kv/(1 - klv),$$
$$v^3\delta_1\kappa = k^{-1}(-2 + 4klv)/(1 - klv) \,.\, \Delta_2 + kv\,(-3 + 4klv)/(1 - klv)^2 \,.\, \Delta_3 \,. \quad . \quad (14)$$

From these equations the values of the curvatures of the two lenses may be found with the help of equations (13). An example of their use will be found below, on p. 44.

In connection with the question of assigning a system which will produce definite changes it may be remarked that it is not difficult to solve the equations (17) of p. 160 of the Memoir so as to give explicitly either $\delta_1 g, \ldots$ or $\delta_1 g', \ldots$ so that we have as may be desired either the antecedent set or the consequent set which combine to produce given aberration coefficients $\delta_1 G, \ldots$. The former are obviously obtained by forming $l'\delta_s G - h'\delta_s K$, $l'\delta_s H - h'\delta_s L$, $-k'\delta_s G + g'\delta_s K$, $-k'\delta_s H + g'\delta_s L$, which give respectively $n'\delta_s g$, $n'\delta_s h$, $n'\delta_s k$, $n'\delta_s l$. For the latter coefficients $\delta_1 g', \ldots$ we form

$$l^2\delta_1 G - 2kl\delta_2 G + k^2\delta_3 G = \ldots + n^2(g\delta_1 g' + k\delta_1 h'),$$
$$l^2\delta_1 H - 2kl\delta_2 H + k^2\delta_3 G = \ldots + n^2(h\delta_1 g' + l\delta_1 h'),$$

which give $\delta_1 g'$, $\delta_1 h'$; and similarly we have $\delta_1 k'$, $\delta_1 l'$. Form also

$$-hl\delta_1\mathrm{G}+(gl+hk)\,\delta_2\mathrm{G}-gk\delta_3\mathrm{G} = \ldots +n^2(g\delta_2 g'+k\delta_2 h'),$$
$$-hl\delta_1\mathrm{H}+\ldots = \ldots +n^2(h\delta_2 g'+l\delta_2 h'),$$

and

$$h^2\delta_1\mathrm{G}-2gh\delta_2\mathrm{G}+g^2\delta_3\mathrm{G} = \ldots +n^2(g\delta_3 g'+k\delta_3 h'),$$
$$h^2\delta_1\mathrm{H}-\ldots = \ldots +n^2(h\delta_3 g'+l\delta_3 h'), \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

with similar equations in $\delta_s\mathrm{K}$, $\delta_s\mathrm{L}$. These equations, for example, answer the question of what aberrations are shown when a known system is reversed and presented with the opposite face to the beam, the unit-points being simply interchanged so that the normal effect as shown in the position of the focus is the same as before. For if an unaberrant beam originating at O is brought to a focus at O′ and shows there aberration coefficients $\delta_1 g, \ldots$; or, what is the same statement, an aberrant beam with coefficients $\delta_1 g, \ldots$ emerging from O′ and passing through the system in the opposite direction is brought to an unaberrant state at O, then if $\delta_1 g', \ldots$ are the coefficients introduced by the reversed passage we have the joint effect of $\delta_1 g', \ldots$ superposed to $\delta_1 g, \ldots$ is null, or $\delta_1\mathrm{G}, \ldots$ are all zero. But it must be noted, as was pointed out for the mirror, that as the direction of the axis is reversed the signs of $\delta_1 k \ldots \delta_3 l$ must be reversed before they are brought into the equations with $\delta_1 g', \ldots$; further, since $\mathrm{G}=1$, $\mathrm{H}=0$, $\mathrm{K}=0$, $\mathrm{L}=1$, we have $g'=l$, $h'=-h$, $k'=-k$, $l'=g$, and $n=1$. The whole question has some general interest, but I shall not pursue it further at present, because it is somewhat beside our mark, and I return to considerations that bear upon the main problem.

Coming now to the immediate object of my paper, which is the Cassegrain telescope, I shall first consider what can be effected with two mirrors simply, which will give opportunities for writing down useful expressions of various forms relating to mirrors.

A mirror with both origins at its surface, and the reversal included, gives the scheme (10) p. 36, or say

$$g=1, \quad h=0, \quad k=k, \quad l=1, \quad p=-k,$$

where $k=2\mathrm{B}$, together with the aberration coefficients

$$\tfrac{1}{2}k^2,\ 0,\ 0;\quad k,\ 0,\ 0;\quad \tfrac{1}{4}(2+\epsilon)\,k^3,\ \tfrac{1}{2}k^2,\ 0;\quad \tfrac{3}{2}k^2,\ k,\ 0. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

With the surface for one origin and the principal focus for the other, these become

$$g=0, \quad h=-k^{-1}, \quad k=k, \quad l=1,$$

with the coefficients

$$-\tfrac{1}{4}\epsilon k^2,\ -\tfrac{1}{2}k,\ 0;\quad -\tfrac{1}{2}k,\ -1,\ 0;\qquad ibid.\qquad;\qquad ibid. \quad . \quad . \quad . \quad . \quad (17)$$

If by the formulæ of the Memoir, p. 164 (22), we transfer the origins to two conjugate foci, P, P′ respectively, say at distances PO = u, OP′ = v along the ray from the surface, so that

$$u+v+kuv = 0$$

—where it is to be noted that the positive direction for both u and v is the direction of the ray, which is reversed at the surface, so that if P, P′ are found upon the same side of the mirror u and v will have the same sign—we have the scheme

$$g = 1+kv, \quad h = 0, \quad k = k, \quad l = 1+ku,$$

with the coefficients

$$\begin{aligned} \delta_s g &= \tfrac{1}{2}k^2[1+kv+\tfrac{1}{2}\epsilon kv], & &\tfrac{1}{4}\epsilon k^3 uv, & &\tfrac{1}{2}k^2uv[1+\tfrac{1}{2}\epsilon ku]; \\ \delta_s h &= k[1+kv+\tfrac{1}{4}\epsilon k^2uv], & &\tfrac{1}{2}k^2uv[1+\tfrac{1}{2}\epsilon ku], & &ku^2[-1+\tfrac{1}{4}\epsilon k^2uv]; \\ \delta_s k &= \tfrac{1}{2}k^3[1+\tfrac{1}{2}\epsilon], & &\tfrac{1}{2}k^2[1+ku+\tfrac{1}{2}\epsilon ku], & &k^2u[1+ku+\tfrac{1}{2}\epsilon ku]; \\ \delta_s l &= \tfrac{1}{2}k^2[3+ku+\tfrac{1}{2}\epsilon ku], & &k+2k^2d+\tfrac{1}{2}k^3d^2(1+\tfrac{1}{2}\epsilon), & &kd[2+\tfrac{5}{2}kd+\tfrac{1}{2}k^2d^2(1+\tfrac{1}{2}\epsilon)]. \end{aligned} \tag{18}$$

To obtain the system for a Cassegrain telescope, we must combine two systems, $(gh\ldots)$, $(g'h'\ldots)$, as in the Memoir, p. 160 (17), of which the former gives the great mirror at its principal focus, by (17) above, while the latter gives the second mirror between two conjugate foci, by (18). Let κ, ϵ refer to the great mirror, and κ', ϵ' to the second one. If we confine attention to spherical aberration, coma, curvature, and astigmatism, it will suffice to form δ_1G, δ_2G, δ_3G for the compound system, deriving δ_2H with the help of the equation $\delta_3\mathrm{G}-\delta_2\mathrm{H} = \mathrm{H}\mathfrak{P}$. The resulting expressions are

$$\begin{aligned} \delta_1\mathrm{G} &= +\tfrac{1}{4}\epsilon\kappa^2 v/u+\kappa^3\kappa' u^2[-1+\tfrac{1}{4}\epsilon'\kappa'^2uv], \\ \delta_2\mathrm{G} &= \tfrac{1}{2}\kappa v/u-\kappa\kappa' u[\kappa u+\tfrac{1}{2}\kappa' v]+\tfrac{1}{4}\epsilon'\kappa\kappa'^3u^2v[-1+\kappa u], \\ \delta_3\mathrm{G} &= -\kappa' v/\kappa u-\kappa' u[\kappa u+\kappa' v]+\tfrac{1}{4}\epsilon'\kappa'^3uv[1-\kappa u]^2/\kappa, \end{aligned} \tag{19}$$

with

$$\delta_3\mathrm{G}-\delta_2\mathrm{H} = -(\kappa+\kappa')\,v/\kappa u.$$

The quantities $\epsilon-1$, $\epsilon'-1$ are what SCHWARZSCHILD calls the deformations of the mirrors, from spherical figures; when $\epsilon = 0$, or the deformation $= -1$, we have a paraboloid; if we choose them so as to annul coma and spherical aberration we have, from the equations $\delta_2\mathrm{G} = 0$, $\delta_1\mathrm{G} = 0$ respectively,

$$\begin{aligned} \tfrac{1}{2}\epsilon'\kappa'^3uv^2 &= 2\kappa' v+1/(1-\kappa u), \\ \epsilon &= -2\kappa u^3/v^2(1-\kappa u); \end{aligned}$$

while if we eliminate ϵ' from δ_3G, we get

$$\begin{aligned} \text{Curvature of field} &= -\mathrm{K}(\delta_3\mathrm{G}+\delta_2\mathrm{H}), \\ &= -\{1+(1-\kappa u)\,\kappa'(v-u)\}/v, \\ &= 1/u+\kappa' u\{1+\kappa(-u+v)\}/v, \end{aligned}$$

and

$$\delta_2\mathrm{H} = -1/\kappa u-(1-\kappa u)/2\kappa v. \tag{20}$$

These expressions are identical, except for notation, with results given by SCHWARZSCHILD; they contain the complete theory of the Cassegrain combination, corrected by figuring for coma and spherical aberration, except as regards distortion, and this could easily be added by calculating δ_3H.

We read from equations (20) that for a given design of instrument, as specified in the values of κ, κ', u or v, we can adjust the figures of the two mirrors so as to annul spherical aberration and coma at the principal focal plane, and then the curvature of the field and astigmatism amount to determinate quantities. Coma is annulled only for the purpose of getting a larger field for photography, and there is very little use in annulling it if the field possesses pronounced curvature, or in less degree, if the focal circles are not reasonably small. Hence the practical questions are: can the design be made such that curvature is nearly absent and astigmatism small, and can the corresponding values assigned to the deformations be realised in practice? All these questions are treated more or less explicitly by SCHWARZSCHILD, and I shall traverse the ground again only in order to connect the problem with its subsequent development and bring out the points which I require.

Regarding the expression for curvature, $v-u$ is the positive distance from the principal focus of the great mirror to the principal focus of the combination. In the Cassegrain form the latter point is, as a rule, not far beyond the surface of the great mirror, so that $v-u$ is not far from the focal length of the great mirror and $1+\kappa(-u+v)$ will be a small fraction; also $\kappa' u$ is numerically less than unity. Hence the curvature of the image will differ very little from $1/u$, the reciprocal of the distance from the second mirror to the principal focus of the great mirror, a distance which would seldom be more than one-third or one-fourth of the focal length of the great mirror, or one-tenth to one-twentieth of the focal length of the combination. The common Cassegrain is subject to the same objection. The values of its errors may be read from the equations (19) on p. 40, if we have the means to determine ϵ, ϵ'.

As an illustration we may take the great 60-inch reflector of Mount Wilson Observatory, which can be used either as a Newtonian, with a focal length of 25 feet, or in three different forms as a Cassegrain; taking the form designed for direct photography, it has an effective focal length of 100 feet, so that $v/u = -4$. If we take the final focus at the great mirror, which is nearly the case, we have $u = -5$, $v = +20$, and $\kappa' = +3/20$. Now since the telescope is corrected as a Newtonian, the great mirror is parabolic, or $\epsilon = 0$; and therefore taking it as corrected for spherical aberration as a Cassegrain, $\frac{1}{4}\epsilon'\kappa'^2 uv = 1$, or $\epsilon' = -16/9$, which is a hyperboloidal form, the deformation from a sphere being nearly three times that which would produce a paraboloid. Substituting $\frac{1}{4}\epsilon'\kappa'^2 uv = 1$ in the equation for δ_2G, we have, after some reductions, $\delta_2\text{G} = \frac{1}{2}\kappa u/v = -\frac{1}{2}\text{K}$, or the coma of such an arrangement is the same as for a simple mirror of the same focal length. Also we find $\delta_3\text{G} = -15$, $\delta_3\text{G}-\delta_2\text{H} = -11$, so that the radius of curvature of the field is one-nineteenth of the focal length or about $5\frac{1}{4}$ feet only. As to the astigmatism

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we have $\delta_2 H = -4$, which may be compared with $\delta_2 H = -1$ for a Newtonian, but since the aperture ratio a/f' is diminished in the ratio 1 : 4 by the increase of effective focal length, the radii of focal circles at all distances from the centre of the field will have the same angular amount that they had in the Newtonian form, neither more nor less. There remains then only the above-found curvature of the field to notice. Taking as a convenient mark a distance 34′·3 from the centre of the field, namely where β in the formulæ of p. 30 equals one-hundredth, we should have at this point the field curved back from the plane through the principal focus by more than one inch. In spite of this pronounced curvature, exquisite photographs of the Moon, as well as of small objects like Mars, have been obtained with this telescope in Cassegrain form. The photograph of the Moon (R.A.S. photographs, No. 214) appears to me second only to the Yerkes photographs with the 40-inch refractor and colour screen; but technically it would be more instructive to examine a photograph of a wide field of stars.

It is worth while to demonstrate that curvature of the field cannot be removed by replacing the second mirror by a set of lenses in contact, used as a reverser, as explained on p. 37. By such a replacement we introduce the quantity $\mathfrak{p}$ which, for a given focal length of the reverser, is adaptable by throwing different proportions of the deviation of the rays upon the lens system and silvered surface respectively.

Then using the formulæ (4) of p. 32, in which we may put $hk = -1$, $l = 1$, k now referring to the great mirror and κ to the reverser,

$$\begin{aligned}
\delta_1 G &= g'\delta_1 g \quad * \quad * \quad + k^3\delta_3 h', \\
\delta_2 G &= g'\delta_2 g \quad * \quad - k\delta_2 h' + k^2\delta_3 h', \\
\delta_3 G &= g'\delta_3 g + k^{-1}\delta_1 h' - 2\delta_2 h' + k\delta_3 h',
\end{aligned}$$

where, if $\delta_1\gamma, \ldots \pi$ refer to the reverser at its surface,

$$\begin{aligned}
\delta_1 h' &= \delta_1\eta + u\delta_1\gamma + v(\delta_1\lambda + u\delta_1\kappa), \\
\delta_2 h' &= \delta_2\eta + \ldots \qquad + u(\delta_1\eta + \ldots), \\
\delta_3 h' &= \delta_3\eta + \ldots \qquad + 2u(\delta_2\eta + \ldots) + u^2(\delta_1\eta + \ldots),
\end{aligned}$$

and by (11), p. 37,

$$\begin{aligned}
\delta_1\eta + \ldots &= \kappa^2 v - \pi(1 + \tfrac{1}{2}\kappa u + \tfrac{1}{2}\kappa v) + uv\delta_1\kappa, \\
\delta_2\eta + \ldots &= -\kappa u + \tfrac{1}{2}uv\kappa\pi, \\
\delta_3\eta + \ldots &= uv(\kappa + \pi).
\end{aligned}$$

Thus

$$\begin{aligned}
\delta_2 G &= g'\delta_2 g + k^2(\delta_3\eta + \ldots) + k(2ku - 1)(\delta_2\eta + \ldots) + ku(ku - 1)(\delta_1\eta + \ldots), \\
\delta_3 G &= g'\delta_3 g + k(\delta_3\eta + \ldots) + 2(ku - 1)(\delta_2\eta + \ldots) + k^{-1}(ku - 1)^2(\delta_1\eta + \ldots).
\end{aligned}$$

Eliminate $(\delta_1\eta + \ldots)$ by forming $(1 - ku)\delta_2 G + k^2 u\delta_3 G$,

$$\begin{aligned}
(1 - ku)\delta_2 G + k^2 u\delta_3 G = g'(1 - ku)\delta_2 g + g'k^2 u\delta_3 g \\
+ k^2(\delta_3\eta + \ldots) - k(1 - ku)(\delta_2\eta + \ldots).
\end{aligned}$$

Now $\delta_2 g = -\frac{1}{2}k$, $\delta_3 g = 0$; and if by figuring or otherwise we annul coma, so that $\delta_2 G = 0$, we have

$$ku\delta_3 G = -\tfrac{1}{2}g'(1-ku)-(1-ku)(-\kappa u+\tfrac{1}{2}uv\kappa\pi)+kuv(\kappa+\pi).$$

Also

$$ku(\delta_3 G-\delta_2 H) = H\mathfrak{P}ku = -(g'/k)(-k+\pi)ku = v(-k+\pi),$$

so that

$$ku(\delta_3 G+\delta_2 H) = -g'(1-ku)+kv+u(1-ku+kv)(2\kappa+\pi),$$

also $K = k/g' = -ku/v$; so that the curvature is

$$-K(\delta_3 G+\delta_2 H) = 1/u+(2\kappa+\pi)(1-ku+kv)u/v. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If we compare this with the expression given in (20) above we see that the sole effect of the change is to replace the reciprocal of the focal length of the second mirror by $(2\kappa+\pi)$ for the reverser, and, since its factor in u, v is small, this change will not allow any considerable modification of the curvature of the field.

To meet the difficulty of curvature SCHWARZSCHILD considers a design of instrument fundamentally altered. Thus in (19) the curvature of the field will vanish if

$$\kappa' = -v/u^2\{1+\kappa(-u+v)\}$$

and this may be secured if κ' is negative as well as κ, or if the second mirror is concave; but in order that the curvature of the mirror may not be too great we must then take $1+\kappa(-u+v)$ sensibly different from zero, and also v/u the magnification of the second mirror, not too large. The system to which SCHWARZSCHILD is led as generally the best to be found under such conditions has been already described (p. 28). It is so different from anything that has yet been made that it must be regarded merely as an interesting exploration of the possibilities of the theory until an attempt is made to realise it. In particular it is utterly different from the long-focus Cassegrain which I have in mind, and therefore I shall not require to refer to it further.

Returning to the question of the Cassegrain proper we see that if an improvement is to be made it must be by inserting a corrector of some form in the course of the beam. Hence we come to the system which I have indicated on p. 29. To get an approximation to what is required, suppose that the reverser is merely a convex mirror, that the corrector consists of a pair of thin lenses of which the theory is given on pp. 37 and 38, and that all the surfaces are spherical except that of the great mirror which is figured so as to annul spherical aberration. To fix ideas I shall suppose that the unit of length is 100 inches, and that with this unit the aperture of the great mirror is 0·40 and its focal length 2·0000, also that the separation of the two mirrors is 1·3333, that the magnification of the second mirror is 2·4, from which it results that its focal length is 1/·875 = 1·1429, and the principal focus of the combination is thrown beyond the great mirror by ·2667, at a distance 1·6000 from the second mirror. It will be seen from the expressions (14) that it is desirable that the corrector should be as far as practicable from the principal focus if its aberrations are to be as small as possible, that is to say, if its curves are to be as shallow as possible. It cannot be too far forward or it will cut off some of the rays coming from the great mirror to the reverser. It appears that a convenient distance is 0·9000 from the reverser, or 0·7000 from the principal focus. That is to say, in the formulæ (19) of p. 40,

$$\kappa = -\cdot 5000, \quad \kappa' = +\cdot 8750, \quad u = -\cdot 6667, \quad v = +1\cdot 6000,$$

so that, with $\epsilon' = 1$, for a spherical reverser,

$$\delta_2 g = +\cdot 3383, \qquad \delta_3 g = -3\cdot 0301.$$

Now we have to make

$$\delta_2 \mathrm{G} = 0, \qquad \delta_3 \mathrm{G} + \delta_2 \mathrm{H} = 0,$$

and we have

$$\delta_3 \mathrm{G} - \delta_2 \mathrm{H} = \mathrm{H}\mathfrak{P} = +4\cdot 8000 \times -\cdot 3750 = -1\cdot 8000.$$

Hence the changes Δ_2, Δ_3, which the corrector must introduce, are respectively,

$$\Delta_2 = -\cdot 3383, \qquad \Delta_3 = +2\cdot 1301.$$

These are the quantities so denoted in (14) p. 38. In the same equation, the values of k, l to be used come from the scheme resulting from the combination of the two mirrors, viz.,

$$g = \quad * \quad, \quad h = +4\cdot 800, \quad k = -\cdot 2083, \quad l = +2\cdot 1667,$$

and v giving the position of the corrector with respect to the principal focus,

$$v = -\cdot 7000.$$

Hence

$$k^{-1}\Delta_2 = +1\cdot 6238, \qquad kv\Delta_3 = +\cdot 3106,$$

$$klv = +\cdot 3160, \quad (1-klv)^{-1} = 1\cdot 4620, \quad (-2+4klv)/(1-klv) = -1\cdot 0760,$$
$$(-3+4klv)/(1-klv)^2 = -3\cdot 7107,$$

and

$$v^2\delta_1\gamma = -1\cdot 6238 - \cdot 4541 = -2\cdot 0779,$$
$$v^3\delta_1\kappa = -1\cdot 7472 - 1\cdot 1525 = -2\cdot 8997.$$

Referring now to (13) for the expressions for $\delta_1\gamma$, $\delta_1\kappa$ for a thin corrector (in which we shall here write κ, κ' in place of k, k'), and remembering that $\mathrm{E} = 0$ since all the surfaces are taken spherical, we have the equations

$$(\kappa v)^2 . \tfrac{1}{2} n [q/\kappa + q'\kappa'] = +2\cdot 0779,$$

$$(\kappa v)^3 n [q/\kappa + q'/\kappa'] \left[1 + \frac{n(1+2n)}{4(1+n)^2}(q/\kappa - q'/\kappa')\right] = -2\cdot 8997.$$

In order to secure shallow curves the quantities q/κ, q'/κ' should be as small as possible. It is therefore evident that κv should be taken negative, that is κ positive. The actual value of κ the reciprocal of the focal length of each member of the corrector has now to be chosen. By increasing κ, q, q' will be made smaller but at the same time the lenses employed will be shortened in focus. As a reasonable trial, take $\kappa = +1{\cdot}4286$, so that $\kappa v = -1$, and the focus of the combination of the two mirrors is also a focus of either lens of the corrector; then taking, say,

$$\mu = 1{\cdot}5200, \quad n = {\cdot}6579, \quad n(1+2n)/4(1+n)^2 = {\cdot}13857,$$

we have

$$q/\kappa + q'/\kappa' = +6{\cdot}3168,$$
$$q/\kappa - q'/\kappa' = -2{\cdot}1809,$$

or the equations give

$$q/\kappa = +2{\cdot}0680, \qquad q'/\kappa' = +4{\cdot}2488.$$

The curvatures of the lenses are now found from

$$\kappa = \left(1-\frac{1}{n}\right)(B_4 - B'_4) = +1{\cdot}4286,$$

$$q = \left(1+\frac{1}{n}\right)(B_4 + B'_4) = +2{\cdot}9543,$$

or

$$B_4 = -{\cdot}7875, \qquad B'_4 = +1{\cdot}9597,$$

and

$$\kappa' = \left(1-\frac{1}{n}\right)(B_6 - B'_6) = -1{\cdot}4286,$$

$$q' = \left(1+\frac{1}{n}\right)(B_6 + B'_6) = -6{\cdot}0697,$$

or

$$B_6 = +{\cdot}1693, \qquad B'_6 = -2{\cdot}5779.$$

These results are a very fair approximation. The final solution, when the thicknesses and consequent separations of all the lenses are allowed for, as well as the introduction of a third weak lens in the reverser to preserve achromatism, with resulting change in the focal length of the second lens of the corrector, is

$$B_4 = -{\cdot}6930, \qquad B'_4 = +2{\cdot}0482,$$
$$B_6 = -{\cdot}0242, \qquad B'_6 = -2{\cdot}6120.$$

The first lens is a double concave, the radii of its two surfaces being 1·270 and 0·510 respectively; the second is double convex, with radii 5·907 and 0·388. The remaining astigmatism is measured by the value of $\delta_2 H$, which by p. 44 is +0·9000, which is about the same as the residual amount present in the focal plane of a refracting doublet. These are all reasonable amounts, so that we are now in possession of a good approximation to a workable solution which corrects coma and curvature of the field, and leaves the figure of the great mirror to correct spherical aberration.

It only remains then to adapt this solution to include consideration of all the secondary factors that have been left on one side.

We must now turn to the question of achromatism in general. A thin corrector, such as is contemplated on p. 37, is, among other properties, achromatic; but when the lenses are made thick and their unit points separated, as must be, to make the system real, this property is lost in greater or less degree. With two lenses only it is not possible to restore it completely. Reserving the quantities q, q' for adjusting aberrations, we may alter the ratio $k : k'$ from the value -1, but this gives only one adjustable element, whereas there are two necessary conditions for achromatism for any specified position of the object, namely, identical position for the image and identical magnification. It is true that in the ordinary achromatised refractor, consisting of a doublet, results are obtained with satisfaction of only a single condition, but the achromatism secured is necessarily very imperfect for another reason—the imperfect rationality of the dispersions of the two kinds of glass—and this masks the neglect of the second condition. For the reflector, where we aim at perfect achromatism, we must add a third lens to supply an additional adjustable element. I shall now give the theory of complete achromatism at a chosen point with three lenses of the same glass, separated by given distances. To make all the lenses of the same glass secures achromatism for all colours if it is attained for any two. The lenses are supposed thin, and the results must therefore be considered merely as approximations, since the thickness will alter the positions of their unit points as well as their focal lengths when a ray of different refractive index is considered. But the approximation will be generally close, and an illustration of how to make a complete adjustment will be given later.

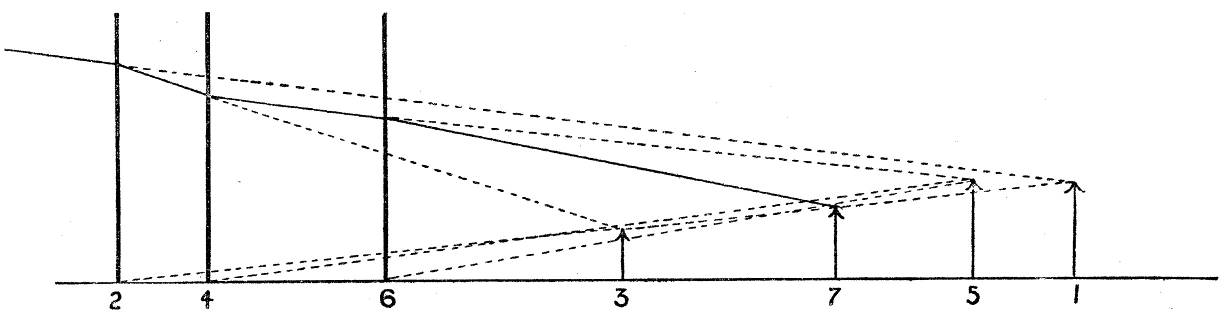

Fig. 1.

Let the lenses be placed at O_2, O_4, O_6 and produce images in succession at P_3, P_5, P_7 of an object at P_1 as shown in the figure. Then the position and size of the image at P_7 must be constant.

Write

$$O_2O_4 = d_3, \quad O_4O_6 = d_5,$$

$$P_1O_2 = v_1, \quad O_2P_3 = u_3; \qquad P_3O_4 = v_3, \quad O_4P_5 = u_5; \qquad P_5O_6 = v_5, \quad O_6P_7 = u_7,$$

also

$$\kappa_2 = \left(1-\frac{1}{n}\right)(B_2-B'_2), \qquad \kappa_4 = \left(1-\frac{1}{n}\right)(B_4-B'_4), \qquad \kappa_6 = \left(1-\frac{1}{n}\right)(B_6-B'_6);$$

then we have the equations

$$v_1+u_3+\kappa_2 v_1 u_3 = 0, \qquad u_3+v_3 = d_3,$$
$$v_3+u_5+\kappa_4 v_3 u_5 = 0, \qquad u_5+v_5 = d_5,$$
$$v_5+u_7+\kappa_6 v_5 u_7 = 0;$$

and the linear magnification is equal to

$$-(u_3/v_1) \,.\, (u_5/v_3) \,.\, (u_7/v_5).$$

Varying the system with respect to $1/n$, the refractive index, and making a condition that v_1, u_7, and the magnification are unchanged, we have

$$\Delta\kappa_2/\kappa_2 = \Delta\kappa_4/\kappa_4 = \Delta\kappa_6/\kappa_6,$$
$$\Delta u_3/u_3^2 = \Delta\kappa_2, \qquad \Delta u_3+\Delta v_3 = 0,$$
$$\Delta v_3/v_3^2+\Delta u_5/u_5^2 = \Delta\kappa_4, \qquad \Delta u_5+\Delta v_5 = 0,$$
$$\Delta v_5/v_5^2 = \Delta\kappa_6,$$

and

$$\Delta u_3/u_3-\Delta v_3/v_3+\Delta u_5/u_5-\Delta v_5/v_5 = 0;$$

eliminate Δu_3, Δv_5 and this gives

$$\Delta v_3\,(1/u_3+1/v_3) = \Delta u_5\,(1/u_5+1/v_5);$$

eliminate Δv_3, Δu_5 and we have the two equations

$$\Delta\kappa_2 \,.\, (u_3 d_3/v_3) = \Delta\kappa_6 \,.\, (v_5 d_5/u_5),$$
$$\Delta\kappa_2 \,.\, (u_3^2/v_3^2)+\Delta\kappa_6 \,.\, (v_5^2/u_5^2) = -\Delta\kappa_4;$$

finally

$$\frac{\kappa_2}{(v_3/u_3 d_3)} = \frac{-\kappa_4}{(u_3/v_3 d_3)+(v_5/u_5 d_5)} = \frac{\kappa_6}{(u_5/v_5 d_5)},$$

or

$$\frac{1/v_1+1/u_3}{1/u_3-1/d_3} = -\frac{1/u_5+1/v_3}{1/u_5+1/v_3-(1/d_3+1/d_5)} = \frac{1/u_7+1/v_5}{1/v_5-1/d_5}, \quad \ldots \quad (23)$$

thus, knowing d_3, d_5, v_1, and choosing, say, κ_2, we determine in succession u_3 the value of the ratio, v_3, u_5, κ_4, v_3, u_7, κ_6. But this choice and order is open to modification.

For example, if we take, as on a subsequent page,

$$d_3 = +\cdot 9261, \quad d_5 = +\cdot 01694, \quad u_3 = +1\cdot 6000, \quad \kappa_4 = +1\cdot 4286,$$

we find

$$\kappa_2 = -\cdot 01056, \qquad \kappa_6 = -1\cdot 3704.$$

This is an illustration of the simple corrector ($\kappa, -\kappa$ in contact) modified by a slight separation of the two lenses and completed by the addition of a weak lens κ_2 at a considerable distance, and adjusted for a point which is nearly at the principal focus of the middle lens. The exact solution on pp. 51–53, gives

$$\kappa_2 = -\cdot 01152, \qquad \kappa_6 = -1\cdot 3459\,;$$

the differences are considerable; this must be expected because the thicknesses of the lenses are of the same order as the separation d_5 of the unit-points; but in all cases the solution will be close enough to supply a good approximation that will allow the actual case to be adjusted.

The general process, suitable for use when we have obtained an approximation by the method just explained, will be the following. Let the standard scheme and that of the varied refractive indices be

$$\{G, \;\; H\,; \;\; K, \;\; L\} \qquad \text{and} \qquad \{G+\Delta G, \;\; H+\Delta H\,; \;\; K+\Delta K, \;\; L+\Delta L\}$$

respectively. Then the conditions for complete achromatism at the principal focus are simply

$$\Delta G = 0, \qquad \Delta K = 0,$$

for these imply that the focal length is unchanged and also the distance $-G/K$ from the origin to the principal focus for either way. Then using the approximation already supposed found, calculate the values of ΔG, ΔK which it shows. Vary the focal length of the first lens and recalculate them. Vary also the third lens and recalculate them. We then have means for interpolating the correct values of the first and third lenses requisite to give an achromatic system in conjunction with the middle lens.

This will be illustrated by the calculation of the actual system which I set out to find and to which I now come. It will be understood that it was obtained by steps of approximation.

It is unnecessary to give details regarding all these steps, which were unnecessarily circuitous, owing to numerical mistakes and ill-judged processes. I shall therefore give the final stage only.

The notation is slightly varied from the standard notation of SEIDEL, O_0 is the vertex of the great mirror, B_0 its curvature, O_2 the vertex of first surface of the reverser, O'_2 the vertex of the second or silvered surface, O''_2, which is the same point as O_2, is the last surface of the reverser; B_2, B'_2, $B''_2 = -B_2$, are the corresponding curvatures; O_4, O'_4 are the vertices of the first and second surfaces of the first lens of the corrector, with curvatures B_4, B'_4; O_6, O'_6 with curvatures B_6, B'_6 refer to the second lens of the corrector. For the thicknesses of the lenses I employ here even suffixes, thus $t_2 = O_2O'_2 = O'_2O''_2$, $t_4 = O_4O'_4$, $t_6 = O_6O'_6$; for the separations, $d_1 = O_0O_2$, $d_3 = O''_2O_4$, $d_5 = O'_4O_6$.

H_2, H''_2 are the unit points of the reverser; H_4, H'_4 and H_6, H'_6, those of the two lenses of the corrector. Similarly F_1 is the principal focus for the great mirror, F_3, F'_3, F''_3 for the different surfaces of the reverser, and so on, the final focus of the whole combination being F'_7.

Writing, as above,

$$\kappa = \left(1-\frac{1}{n}\right)(B-B'), \qquad q = \left(1+\frac{1}{n}\right)(B+B'), \qquad \mathfrak{q} = q/K,$$

we find by considering the scheme

$$\begin{Bmatrix} 1, & * \\ (n-1)\,B, & n \end{Bmatrix}\begin{Bmatrix} 1, & t \\ * & 1 \end{Bmatrix}\begin{Bmatrix} 1, & * \\ (n^{-1}-1)\,B', & n^{-1} \end{Bmatrix} \equiv \begin{Bmatrix} G, & H \\ K, & L \end{Bmatrix},$$

that for any thick lens

$$K = \kappa - n\,(1-n^{-1})^2\, tBB' = \kappa + \tfrac{1}{4}nt\kappa^2 - \tfrac{1}{4}n\,(1-n)^2\,(1+n)^{-2}\,tq^2,$$

and

$$OH = (L-1)/K = (n^{-1}-1)\,tB'/K, \qquad O'H' = (1-G)/K = (1-n)\,tB/K. \quad . \quad (24)$$

For the reverser we have the scheme, including reversal of the ray at the reflection,

$$\begin{Bmatrix} 1, & * \\ (n-1)\,B_2, & n \end{Bmatrix}\begin{Bmatrix} 1, & t_2 \\ * & 1 \end{Bmatrix}\begin{Bmatrix} 1, & * \\ 2B'_2, & 1 \end{Bmatrix}\begin{Bmatrix} 1, & t_2 \\ * & 1 \end{Bmatrix}\begin{Bmatrix} 1, & * \\ -(n^{-1}-1)\,B_2, & n^{-1} \end{Bmatrix} \equiv \begin{Bmatrix} G_2, & H_2 \\ K_2, & L_2 \end{Bmatrix}$$

whence

$$K_2 = 2n^{-1}k_2\,(1+t_2k_2) + 2n^{-1}B'_2\,(1+t_2k_2)^2, \qquad k_2 = (n-1)\,B_2,$$
$$O_2H_2 = nt_2/(1+t_2k_2) = -O''_2H''_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Write (K_2) for the part of K_2 which is due to the lens of the reverser, namely,

$$(K_2) = (1-n^{-1})\,(B_2-B'_2) - n\,(1-n^{-1})^2\,t_2B_2B'_2.$$

By methods essentially the same as those exposed below I was led to the following approximate values as a system corrected for aberrations :—

$$B_0 = -\cdot 250000, \quad \epsilon_0 = +\cdot 16502, \quad \alpha_0 = +\cdot 200000,$$
$$d_1 = +1\cdot 320133,$$
$$B_2 = -B''_2 = +\cdot 469009, \quad B'_2 = +\cdot 450653, \quad t_2 = +\cdot 020000,$$
$$d_3 = +\cdot 906760,$$
$$B_4 = -\cdot 697845, \quad B'_4 = +2\cdot 043309, \quad t_4 = +\cdot 012500,$$
$$d_5 = +\cdot 002500,$$
$$B_6 = +\cdot 003705, \quad B'_6 = -2\cdot 610677, \quad t_6 = +\cdot 012500.$$

The initial semi-aperture, a_0, does not enter the calculations, but is carried through at its adopted value, which is recorded here for reference.

It follows that

$$K_2 = +{}^{\cdot}875000, \qquad (K_2) = -{}^{\cdot}010297,$$

$$O_2H_2 = -O''_2H''_2 = +{}^{\cdot}013200,$$

$$K_4 = +1{}^{\cdot}428571, \quad O_4H_4 = +{}^{\cdot}006116, \quad O'_4H'_4 = -{}^{\cdot}002089,$$

$$K_6 = -1{}^{\cdot}359456, \quad O_6H_6 = +{}^{\cdot}008211, \quad O'_6H'_6 = -{}^{\cdot}000012,$$

and that

$$O_0H_2 = +1{}^{\cdot}333333, \qquad H''_2F''_3 = +1{}^{\cdot}600000, \quad \ldots \quad (26)$$

and the power of the combination of great mirror and reverser is the same as in the preliminary solution. The achromatism of the system proved also satisfactory, but the numbers had to be recast because of the following defect. As will be seen on p. 63, the semi-aperture of the lenses of the corrector is about $a = +{}^{\cdot}0615$. Hence the separations of the vertices of the surfaces which are next to one another must be at least $\frac{1}{2}a^2(B'_4 - B_6) = +{}^{\cdot}00387$. Hence enough separation has not been allowed, since we have taken $d_5 = +{}^{\cdot}00250$. I therefore increased d_5 to the value of $+{}^{\cdot}005000$. At the same time I decided to increase the thickness t_6 also to $t_6 = +{}^{\cdot}015000$. To change d_5, t_6 means upsetting the balance of achromatism between the lens of the reverser and the lenses of the corrector. All the quantities then will require adjustment. The first step is to re-establish the achromatism. In doing so I keep the first lens of the corrector unchanged, and two trials at least will be requisite to get material for a proper adjustment of the other two as explained on p. 48. I found by inspection and by previous trials that an alteration of the second lens of the corrector produces its effect almost solely upon the coefficient K of the final scheme, and hardly at all upon G; hence I first adjust the lens of the reverser so as to make $\Delta G = 0$ for variation of refractive index, and then the second lens of the corrector so as to make $\Delta K = 0$ also. Since the system $O''_2 \ldots O_6$ from the last face of the reverser to the first face of the second lens is unaltered throughout I take it in one piece, taking the lens (4) with the data of p. 49, and

$$d_3 = +{}^{\cdot}906760, \qquad d_5 = +{}^{\cdot}005000,$$

and taking in succession

$$\mu = n^{-1} = 1{}^{\cdot}520000, \qquad n = {}^{\cdot}657895$$

and

$$\mu + \delta\mu = (n + \delta n)^{-1} = 1{}^{\cdot}01 \times n^{-1} = 1{}^{\cdot}535200. \qquad n + \delta n = {}^{\cdot}651381,$$

we have then for the piece $O''_2 \ldots O_6$,

$$\left\{\begin{matrix} 1, & +{\cdot}906760 \\ * & 1 \end{matrix}\right\}\left\{\begin{matrix} 1, & * \\ +{\cdot}238736, & +{\cdot}657895 \end{matrix}\right\}\left\{\begin{matrix} 1, & +{\cdot}012500 \\ * & 1 \end{matrix}\right\}\left\{\begin{matrix} 1, & * \\ +1{\cdot}062521, & +1{\cdot}520000 \end{matrix}\right\}\times$$

$$\times\left\{\begin{matrix} 1, & +{\cdot}005000 \\ * & 1 \end{matrix}\right\} = \left\{\begin{matrix} +1{\cdot}010127, & +\;{\cdot}929210 \\ +1{\cdot}428571, & +2{\cdot}304108 \end{matrix}\right\} \quad [n],$$

and

$$\left\{\begin{matrix} 1, & +{\cdot}906760 \\ * & 1 \end{matrix}\right\}\left\{\begin{matrix} 1, & * \\ +{\cdot}243282, & +{\cdot}651381 \end{matrix}\right\}\left\{\begin{matrix} 1, & +{\cdot}012500 \\ * & 1 \end{matrix}\right\}\left\{\begin{matrix} 1, & * \\ +1{\cdot}093579, & +1{\cdot}535200 \end{matrix}\right\}\times$$

$$\times\left\{\begin{matrix} 1, & +{\cdot}005000 \\ * & 1 \end{matrix}\right\} = \left\{\begin{matrix} +1{\cdot}010393, & +\;{\cdot}929370 \\ +1{\cdot}470392, & +2{\cdot}342197 \end{matrix}\right\} \quad [n+\delta n].$$

Now, taking the reverser first as given on p. 49, we have for the system $O_0 \ldots O''_2$,

$$\left\{\begin{matrix} 1, & * \\ -{\cdot}500000, & 1 \end{matrix}\right\}\left\{\begin{matrix} 1, & +1{\cdot}320133 \\ * & 1 \end{matrix}\right\}\left\{\begin{matrix} 1, & * \\ -{\cdot}160450, & +{\cdot}657895 \end{matrix}\right\}\left\{\begin{matrix} 1, & +{\cdot}020000 \\ * & 1 \end{matrix}\right\}\times$$

$$\times\left\{\begin{matrix} 1, & * \\ +{\cdot}901306, & 1 \end{matrix}\right\}\left\{\begin{matrix} 1, & +{\cdot}020000 \\ * & 1 \end{matrix}\right\}\left\{\begin{matrix} 1, & * \\ -{\cdot}243885, & +1{\cdot}520000 \end{matrix}\right\}$$

$$= \left\{\begin{matrix} +{\cdot}330582, & +1{\cdot}361934 \\ -{\cdot}208333, & +2{\cdot}16667 \end{matrix}\right\} \quad [n],$$

and

$$\left\{\begin{matrix} 1, & * \\ -{\cdot}500000, & 1 \end{matrix}\right\}\left\{\begin{matrix} 1, & +1{\cdot}320133 \\ * & 1 \end{matrix}\right\}\left\{\begin{matrix} 1, & * \\ -{\cdot}163505, & +{\cdot}651381 \end{matrix}\right\}\left\{\begin{matrix} 1, & +{\cdot}020000 \\ * & 1 \end{matrix}\right\}\times$$

$$\times\left\{\begin{matrix} 1, & * \\ +{\cdot}901306, & 1 \end{matrix}\right\}\left\{\begin{matrix} 1, & +{\cdot}020000 \\ * & 1 \end{matrix}\right\}\left\{\begin{matrix} 1, & * \\ -{\cdot}251014, & +1{\cdot}535200 \end{matrix}\right\}$$

$$= \left\{\begin{matrix} +{\cdot}330672, & +1{\cdot}361508 \\ -{\cdot}208522, & +2{\cdot}165574 \end{matrix}\right\} \quad [n+\delta n],$$

also the second lens of the corrector $O_6 \ldots O'_6$ with curvatures, as given on p. 49, but increasing the thickness to $\cdot 015000$, is

$$\left\{\begin{matrix} 1, & * \\ -\cdot 001267, & +\cdot 657895 \end{matrix}\right\} \left\{\begin{matrix} 1, & +\cdot 015000 \\ * & 1 \end{matrix}\right\} \left\{\begin{matrix} 1, & * \\ -1\cdot 357552, & +1\cdot 520000 \end{matrix}\right\}$$

$$= \left\{\begin{matrix} +\cdot 999981 & +\cdot 009868 \\ -1\cdot 359452 & +\cdot 986604 \end{matrix}\right\} \qquad [n],$$

and

$$\left\{\begin{matrix} 1, & * \\ -\cdot 001292, & +\cdot 651381 \end{matrix}\right\} \left\{\begin{matrix} 1, & +\cdot 015000 \\ * & 1 \end{matrix}\right\} \left\{\begin{matrix} 1, & * \\ -1\cdot 397234, & 1\cdot 535200 \end{matrix}\right\}$$

$$= \left\{\begin{matrix} +\cdot 999981, & +\cdot 009771 \\ -1\cdot 399190, & +\cdot 986348 \end{matrix}\right\} \qquad [n+\delta n].$$

Hence the whole combination gives

$$\left\{\begin{matrix} \cdot 330582, & +1\cdot 361934 \\ -\cdot 208333, & +2\cdot 166667 \end{matrix}\right\} \left\{\begin{matrix} +1\cdot 010127, & +\cdot 929210 \\ +1\cdot 428571, & +2\cdot 304108 \end{matrix}\right\} \left\{\begin{matrix} +\cdot 999981, & +\cdot 009868 \\ -1\cdot 359452, & +\cdot 986604 \end{matrix}\right\}$$

$$= \left\{\begin{matrix} +\cdot 140265, & +3\cdot 457413 \\ -\cdot 198450, & +2\cdot 237713 \end{matrix}\right\} \qquad [n]$$

and

$$\left\{\begin{matrix} \cdot 330672, & +1\cdot 361508 \\ -\cdot 208522, & +2\cdot 165574 \end{matrix}\right\} \left\{\begin{matrix} +1\cdot 010393, & +\cdot 929370 \\ +1\cdot 470392, & +2\cdot 342197 \end{matrix}\right\} \left\{\begin{matrix} +\cdot 999981, & +\cdot 009771 \\ +1\cdot 399190, & +\cdot 986348 \end{matrix}\right\}$$

$$= \left\{\begin{matrix} +\cdot 140291, & +3\cdot 457336 \\ -\cdot 198480, & +2\cdot 236730 \end{matrix}\right\} \qquad [n+\delta n].$$

Hence for $n+\delta n$ there is an excess in the coefficient G of 26 units; to correct this, guided by previous experiments, I made a trial change in (K_2), which refers to the lens of the reverser, of -1220 units, so that

$$(K_2) = -\cdot 010297 - \cdot 001220 = -\cdot 011517.$$

This gives, to redetermine the reverser, supposing its power is to remain unchanged,

$$2n^{-1}k_2(1+t_2k_2) + 2n^{-1}B'_2(1+t_2k_2)^2 = +\cdot 875000,$$

$$k_2 = (n-1)B_2, \qquad t_2 = +\cdot 020000,$$

$$(1-n^{-1})(B_2 - B'_2) - n(1-n^{-1})^2 t_2 B_2 B'_2 = -\cdot 011517,$$

whence

$$B_2 = +\cdot 472584, \qquad B'_2 = +\cdot 451898,$$

and these give for the system $O_0 \ldots O''_2$ the schemes, built up just as on the previous page,

$$[n], \quad \begin{Bmatrix} +\cdot 330582, & +1\cdot 361934 \\ -\cdot 208333, & +2\cdot 166667 \end{Bmatrix} \quad \text{and} \quad [n+\delta n], \quad \begin{Bmatrix} +\cdot 330671, & +1\cdot 361508 \\ -\cdot 208547, & +2\cdot 165477 \end{Bmatrix}$$

of which the first is the same as we had before, supplying a verification of the solution of the equations for B_2, B'_2.

Substitute these in the schemes $O_0 \ldots O'_6$ in place of the values already used; $[n]$ is, of course, unchanged, and we find for

$$[n+\delta n], \quad \begin{Bmatrix} +\cdot 140266, & +3\cdot 457242 \\ -\cdot 198505, & +2\cdot 236634 \end{Bmatrix}.$$

Hence G has now the same value in both schemes and it is unnecessary to make a further trial or change of the reverser, but there remains an excess in K of -55 units; to deal with this, try reducing the curvature of each face of the second lens of the corrector by one-hundredth part. This will give the schemes

$$\begin{Bmatrix} 1, & * \\ -\cdot 001254, & +\cdot 657895 \end{Bmatrix} \begin{Bmatrix} 1, & +\cdot 015000 \\ * & 1 \end{Bmatrix} \begin{Bmatrix} 1, & * \\ -1\cdot 343976, & +1\cdot 520000 \end{Bmatrix}$$

$$= \begin{Bmatrix} +\ \cdot 999981, & +\cdot 009868 \\ -1\cdot 345856, & +\cdot 986738 \end{Bmatrix} \quad [n],$$

and

$$\begin{Bmatrix} 1, & * \\ -\cdot 001279, & +\cdot 651381 \end{Bmatrix} \begin{Bmatrix} 1, & +\cdot 015000 \\ * & 1 \end{Bmatrix} \begin{Bmatrix} 1, & * \\ -1\cdot 383262, & +1\cdot 535200 \end{Bmatrix}$$

$$= \begin{Bmatrix} +\ \cdot 999981, & +\cdot 009771 \\ -1\cdot 385200, & +\cdot 986484 \end{Bmatrix} \quad [n+\delta n].$$

Substituting these in the combination $O_0 \ldots O'_6$ we get

$$[n], \quad \begin{Bmatrix} +\cdot 140265, & +3\cdot 457414 \\ -\cdot 196543, & +2\cdot 284718 \end{Bmatrix} \quad \text{and} \quad [n+\delta n], \quad \begin{Bmatrix} +\cdot 140266, & +3\cdot 457242 \\ -\cdot 196543, & +2\cdot 284996 \end{Bmatrix} \qquad (27)$$

Hence both G and K are now identical, and, in consequence, both schemes indicate the same principal focus and the same focal length; in other words, complete achromatism at the principal focus.

We now return to the aberrations; we have replaced the numbers of p. 49 by the following:—

$$\begin{aligned}
&B_0 = -\ {\cdot}250000,\\
&d_1 = +1{\cdot}320133,\\
&B_2 = -B''_2 = +{\cdot}472584, \quad B'_2 = +{\cdot}451898, \quad t_2 = +{\cdot}020000,\\
&d_3 = +{\cdot}906760,\\
&B_4 = -{\cdot}697845, \quad B'_4 = +2{\cdot}043309, \quad t_4 = +{\cdot}012500,\\
&d_5 = +{\cdot}005000,\\
&B_6 = +{\cdot}003667, \quad B'_6 = -2{\cdot}584570, \quad t_6 = +{\cdot}015000, \quad \ldots \ldots \quad (28)
\end{aligned}$$

and in these changes the aberrations calculated for the lenses of p. 49 will be changed; we now require to find new values for q_4, q_6, which will restore the disturbed correction. It may be remarked that the chromatic correction depends very little upon the distribution of the curvatures between the two faces which is indicated in the value of q, and it might have been reflected that as the surface (6) is nearly plane, and the beam meets it nearly at right angles, while the surface (6′) produces almost the whole deviation of the beam for which the second lens is answerable, it would have been better to keep B'_6 unmodified while the second lens was adjusted for achromatism, but this was not noticed until the solution which follows had been made, and was found to reproduce almost exactly the value of B'_6 of p. 49.

The aberration coefficients for a thin lens at its surface are given by (6), p. 33. I have not so far succeeded in supplementing these by any algebraic expression containing reference to thicknesses or separations of lenses, which are simple enough to be useful. Hence the procedure for finding q_4, q_6, ϵ_0 must be by approximation, and the following is the method adopted. Calculate at the principal focus of the complete combination given by (28) the numerical values of the aberration coefficients, or at least the essential ones $\delta_1 G$, $\delta_2 G$, $\delta_3 G$, in three parts, namely, first, the great mirror and reverser together in which ϵ_0 is easily included as an unknown; second, the first lens of the corrector; and third, the second lens of the corrector. The conditions for a corrected system are then

$$\delta_1 G = 0, \quad \delta_2 G = 0, \quad \delta_3 G = \tfrac{1}{2} H\mathfrak{P};$$

supposing these are not satisfied we must bring in corrected values of q_4, q_6, ϵ_0 to satisfy them. I assume for the purpose of approximate correction that the quantities q, q^2 enter the calculated aberrations with the same coefficients as if the lenses were thin; on this supposition I calculate the algebraic values of the aberrations, carrying them from the surfaces of the lenses forwards to F'_7 and backwards to O_0 by a double application of the formulæ (17) of p. 160 of the Memoir. Assuming that these expressions involving the adjustable parameters ϵ_0, q_4, q_6 account for the discrepancies we have equations to determine ϵ_0, q_4, q_6, and in consequence amended values of the curvatures of (28), that is to say, the material to repeat the approximation, if required, and finally to prove that no further change is necessary.

The numerical calculation of all aberrations follows the model given in the Memoir, pp. 172 *et seq.*, and it will be unnecessary as a rule to give details of the working here, though I may mention that I have found a noteworthy abbreviation of it.

The great mirror and reverser together, the former treated as parabolic, contribute at F'_7

$$\delta_1 G = \ldots +{\cdot}057176, \quad \delta_2 G = +{\cdot}354860, \quad \delta_3 G = -3{\cdot}297523.$$

We must also introduce the deviation of the mirror from a paraboloid, viz., we have at the surface of the mirror the additional term $\delta_1 k = \ldots + 2\epsilon_0 B_0{}^3 = -{\cdot}0312500\epsilon_0$, and all the others unaffected. To find the effect of this in the final set $\delta_1 G, \ldots$, by (17) of the Memoir we must take $h'\delta_1 k$ in $\delta_1 G$ merely, where h' belongs to the scheme $O_0 \ldots F'_7$ and is simply equal to the final focal length, which comes out $+5{\cdot}087942$; hence we must supplement the numerical values above by the unknown term

$$\delta_1 G = \ldots -{\cdot}158998\epsilon_0, \quad \delta_2 G = *, \quad \delta_3 G = *.$$

Next we find that the other two lenses contribute together at F'_7

$$\delta_1 G = \ldots -{\cdot}030167, \quad \delta_2 G = \ldots -{\cdot}342260, \quad \delta_3 G = \ldots +2{\cdot}494419.$$

Further, for the three sections

$$\mathfrak{P} = -{\cdot}417950 + {\cdot}937763 - {\cdot}885449 = -{\cdot}365636,$$
$$H = +5{\cdot}087942, \qquad \tfrac{1}{2}\mathfrak{P}H = -{\cdot}930167,$$

and the three equations to satisfy being

$$\delta_1 G = 0, \qquad \delta_2 G = 0, \qquad \delta_3 G = -{\cdot}930167,$$

we find the actual numbers leave residuals in the left-hand members of the values

$$+{\cdot}027009, \qquad +{\cdot}012600, \qquad +{\cdot}127057. \quad \ldots \quad (29)$$

These are to be brought to zero when supplemented by the proper expressions in ϵ_0, q_4, q_6, and ϵ_0 is dealt with above.

Now referring to the expressions for a thin lens and writing $\mathfrak{q} = q/k$ so that for the system just computed $\mathfrak{q}_4 = +2{\cdot}3734$, $\mathfrak{q}_6 = +4{\cdot}8325$, and confining attention to the forms in which $\mathfrak{q}$ is introduced at the surfaces of the lenses, these are respectively :—

First lens—

$$\delta_1 \gamma_4 = \ldots -{\cdot}671321\mathfrak{q}_4,$$
$$\delta_1 \kappa_4 = \ldots +{\cdot}959031\mathfrak{q}_4 + {\cdot}265793\mathfrak{q}_4{}^2, \qquad \delta_2 \kappa_4 = \ldots +{\cdot}671321\mathfrak{q}_4,$$
$$\delta_1 \lambda_4 = \ldots +{\cdot}671321\mathfrak{q}_4,$$

and the rest zero ;

Second lens—

$$\delta_1\gamma_6 = \ldots -{\cdot}595833\mathfrak{q}_6,$$

$$\delta_1\kappa_6 = \ldots -{\cdot}801905\mathfrak{q}_6 - {\cdot}222246\mathfrak{q}_6^2, \qquad \delta_2\kappa_6 = \ldots +{\cdot}595833\mathfrak{q}_6,$$

$$\delta_1\lambda_6 = \ldots +{\cdot}595833\mathfrak{q}_6, \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and the rest zero.

For the second lens, the subsequent normal scheme $O'_6 \ldots F'_7$ is

$$\{g', h' ;\ k' l'\} = \{1, +{\cdot}713667 ;\quad * \quad, 1\}$$

and by (17) of the Memoir, this gives for the second lens from $O_6 \ldots F'_7$ the terms in q :—

	Coefficient, $\mathfrak{q}_6$.	Coefficient, $\mathfrak{q}_6^2$.
$\delta_1\gamma_6 =$	$-1{\cdot}168126$	$-{\cdot}158610$
$\delta_2\gamma_6 =$	$+\ {\cdot}425226$	*
$\delta_3\gamma_6 =$	*	*
$\delta_1\eta_6 =$	$+\ {\cdot}425226$	*
$\delta_2\eta_6 =$	*	*
$\delta_3\eta_6 =$	*	*

The preceding normal scheme $O_0 \ldots O_6$ is

$$\{g, h ;\ k, l\} = \{+{\cdot}140346, +3{\cdot}389017 ;\ -{\cdot}007760, +6{\cdot}937860\}.$$

We see, by referring to the equations (17) of the Memoir already quoted so frequently, that in order to get $\delta_1 G$, $\delta_2 G$, $\delta_3 G$, we must form $g\delta_s\gamma_6 + k\delta_s\eta_6$ $(s = 1, 2, 3)$ with these values of g, h, k, l, and multiplying them respectively by

$$g^2 = +\ {\cdot}019697, \qquad 2gk = -\ {\cdot}002178, \qquad k^2 = +\ {\cdot}000060,$$

$$gh = +\ {\cdot}475635, \qquad gl + hk = +\ {\cdot}947402, \qquad kl = -\ {\cdot}053838,$$

$$h^2 = +11{\cdot}485436, \qquad 2hl = +47{\cdot}025051, \qquad l^2 = +48{\cdot}133901,$$

take the sums. The values of $g\delta_s\gamma_6 + k\delta_s\eta_6$ are

	Coefficient, $\mathfrak{q}_6$.	Coefficient, $\mathfrak{q}_6^2$.
$s = 1$	$-{\cdot}167242$	$-{\cdot}022260$
2	$+{\cdot}059679$	*
3	*	*

the resulting values are

	Coefficient, $\mathfrak{q}_6$.	Coefficient, $\mathfrak{q}_6^2$.	
$\delta_1 G =$	$-{\cdot}003424$	$-{\cdot}000438{,}5$	
$\delta_2 G =$	$-{\cdot}023006$	$-{\cdot}010587{,}6$	
$\delta_3 G =$	$+{\cdot}885540$	$-{\cdot}255667{,}4$	(31)

In the same way, for the first lens of the corrector, the subsequent normal scheme $O'_4 \ldots O_7$ is $\{g', h'; k', l'\} = \{+\cdot039488, +\cdot714268; -1\cdot345856, +\cdot980009\}$, which gives for the first system, between $O_4 \ldots F'_7$

	Coefficient, $\mathfrak{q}_4$.	Coefficient, $\mathfrak{q}_4^2$.
$\delta_1\gamma_4 =$	$+\cdot658496$	$+\cdot189847$
$\delta_2\gamma_4 =$	$+\cdot479503$	*
$\delta_3\gamma_4 =$	*	*
$\delta_1\eta_4 =$	$+\cdot479503$	*
$\delta_2\eta_4 =$	*	*
$\delta_3\eta_4 =$	*	*

The preceding normal scheme $O_0 \ldots O_4$ is

$$\{g, h; k, l\} = \{+\cdot141675, +3\cdot326584; -\cdot208333, +2\cdot166667\},$$

which gives

$$g^2 = +\ \cdot020072, \quad 2gk = -\ \cdot059031, \quad k^2 = +\ \cdot043403,$$
$$gh = +\ \cdot471294, \quad gl+hk = -\ \cdot386074, \quad kl = -\ \cdot451388,$$
$$h^2 = +11\cdot066161, \quad 2hl = +14\cdot415196, \quad l^2 = +4\cdot694446,$$

so that with the values of $g\delta_s\gamma_4 + k\delta_s\eta_4$, which are

	Coefficient, $\mathfrak{q}_4$.	Coefficient, $\mathfrak{q}_4^2$.
$s = 1$	$-\cdot006604$	$+\cdot026897$
2	$+\cdot067934$	*
3	*	*

we find the contributions of the first lens $O_0 \ldots F'_7$

	Coefficient, $\mathfrak{q}_4$.	Coefficient, $\mathfrak{q}_4^2$.	
$\delta_1 G =$	$-\cdot004143$	$+\cdot000539,9$	
$\delta_2 G =$	$-\cdot029340$	$+\cdot012676,2$	
$\delta_3 G =$	$+\cdot906201$	$+\cdot297642,1$	(32)

With the values $\mathfrak{q}_4 = +2\cdot3734$, $\mathfrak{q}_6 = +4\cdot8325$, the joint contribution of the two lenses in respect to the terms $\mathfrak{q}$, $\mathfrak{q}^2$ would be, from these expressions,

$$\delta_1 G = \ldots -\cdot033578, \qquad \delta_2 G = \ldots -\cdot356618, \qquad \delta_3 G = \ldots +2\cdot136186.$$

Hence if new values of ϵ_0, $\mathfrak{q}_4$, $\mathfrak{q}_6$ are to satisfy the conditions exactly, these are determined by the equations

	Coefficient, ϵ_0.	Coefficient, $\mathfrak{q}_4$.	Coefficient, $\mathfrak{q}_4^2$.	Coefficient, $\mathfrak{q}_6$.	Coefficient, $\mathfrak{q}_6^2$.	Constant.	
0 =	$-\cdot158998$	$-\cdot004143$	$+\cdot000539,9$	$-\cdot003424$	$-\cdot000438,5$	$+\ \cdot060587$	
0 =	*	$-\cdot029340$	$+\cdot012676,2$	$-\cdot023006$	$-\cdot010587,6$	$+\ \cdot369218$	
0 =	*	$+\cdot906201$	$+\cdot297642,1$	$+\cdot885540$	$-\cdot255667,4$	$-2\cdot009113$.	(33)

the solutions of which are

$$\mathfrak{q}_4 = +2\cdot390547, \qquad \mathfrak{q}_6 = +4\cdot936038, \qquad \epsilon_0 = +\cdot164675. \quad . \quad . \quad (33\text{A})$$

If with these values of $\mathfrak{q}_4$, $\mathfrak{q}_6$ we calculate the curvatures of the two lenses from the formulæ (24) of p. 49, we find that the completed approximation directs us to replace the numbers of p. 54 from which we set out by

$$B_4 = -\cdot693009, \qquad B'_4 = +2\cdot048193,$$
$$B_6 = -\cdot024163, \qquad B'_6 = -2\cdot612025, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

together with the value of ϵ_0 just written down.

Turning back to p. 49, where these data from a previous approximation are set down, we see that the chief effect of the step is to restore B'_6 to the value given on p. 49, throwing the change in focal length which is demanded for achromatism, in accordance with p. 53, almost exclusively upon B_6, which is a surface that contributes very little to these aberrations. The changes are thus in reality smaller than they appear. Following now strictly the plan given on p. 54, the next step is to take the new system as a whole and calculate exactly its numerical aberrations at its principal focus; it is unnecessary to give the details of this step, which contains nothing new; the following numbers show first the normal schemes from the surface O_0 up to each other point, and then the contribution of each surface to each of the coefficients $\delta_1 G \ldots \delta_3 H$ at the principal focus F'_7.

$$\mu = 1\cdot5200.$$

Normal Schemes, $\{g, h\,;\ k, l\}$.

Surface O_0	{+1·000000,	* ;	−·500000,	+1·000000},
O_0 to O_2	{+ ·339933,	+1·320133 ;	−·500000,	+1·000000},
O_0 to O_2 and surface O_2	{+ ·339933,	+1·320133 ;	−·383906,	+ ·444465},
O_0 to O'_2	{+ ·332255,	+1·329022 ;	−·383906,	+ ·444465},
O_0 to O'_2 and surface O'_2	{+ ·332255,	+1·329022 ;	−·083615,	+1·645630},
O_0 to O''_2	{+ ·330583,	+1·361935 ;	−·083615,	+1·645630},
O_0 to O''_2 and surface O''_2	{+ ·330583,	+1·361935 ;	−·208333,	+2·166667},
O_0 to O_4	{+ ·141675,	+3·326584 ;	−·208333,	+2·166667},
O_0 to O_4 and surface O_4	{+ ·141675,	+3·326584 ;	−·103472,	+2·214112},
O_0 to O'_4	{+ ·140382,	+3·354260 ;	−·103472,	+2·214112},
O_0 to O'_4 and surface O'_4	{+ ·140382,	+3·354260 ;	−·007762,	+6·937938},
O_0 to O_6	{+ ·140343,	+3·388950 ;	−·007762,	+6·937938},
O_0 to O_6 and surface O_6	{+ ·140343,	+3·388950 ;	−·003947,	+4·592448},
O_0 to O'_6	{+ ·140284,	+3·457837 ;	−·003947,	+4·592448},
O_0 to O'_6 and surface O'_6	{+ ·140284,	+3·457837 ;	−·196540,	+2·283904},
O_0 to F'_7	{ * ,	+5·088015 ;	−·196540,	+2·283904}.

. . . . . . (35)

$\mu = 1·5200.$

Aberration Coefficients at F'_7.

Surface.	Lateral.	Obliquity.	Lateral.	Obliquity.	Lateral.	Obliquity.
0	+·29182	−·31800	*	+·63600	*	*
2	−·01094	+·06023	−·04251	−·19332	− ·16509	+ ·47099
2′	−·04216	+·08433	−·16866	−·01984	− ·67464	− 1·50801
2″	−·01195	−·02233	−·04922	+·19242	− ·20279	− 1·21803
4	+·00227	+·00722	+·05341	−·04912	+ 1·25414	− ·04592
4′	+·00550	−·00212	+·13140	+·03502	+ 3·13954	+ 8·63910
6	·00000	·00000	·00000	−·00029	+ ·00006	+ ·21537
6′	−·02836	−·01552	−·69898	+·17309	−17·22893	+ 6·37874
	+·20618	−·20619	−·77456	+·77396	−13·87771	+12·93224
	$\delta_1 G = -·00001$		$\delta_2 G = -·00060$		$\delta_3 G = -·94547$	

Surface.	Lateral.	Obliquity.	Lateral.	Obliquity.	Lateral.	Obliquity.
0	−·63600	+1·27200	*	−2·54401	*	*
2	+·05238	− ·28821	+ ·20342	+ ·92507	+ ·79000	− 2·25380
2′	+·18850	− ·37701	+ ·75400	+ ·08871	+ 3·01601	+ 6·74169
2″	+·04992	+ ·09328	+ ·20564	− ·80386	+ ·84719	+ 5·08862
4	+·00103	+ ·00327	+ ·02417	− ·02222	+ ·56746	− ·02078
4′	+·27142	− ·10500	+6·48513	+1·72840	+154·95440	+426·39006
6	−·00006	− ·00021	− ·00153	+ ·17490	− ·03694	−132·40118
6′	−·33991	− ·18598	−8·37835	+2·07474	−206·51664	+ 76·45958
	−·41272	+ ·41214	− ·70752	+1·62173	− 46·37852	+380·00419
	$\delta_1 H = -·00058$		$\delta_2 H = +·91421$		$\delta_3 H = +333·62567$	

. . . . (36)

From these results we read the particulars of the field from the data given on p. 30. We have, by pp. 58, 49,

$$f' = +5·088015, \qquad a = +·200000.$$

Hence, from $\delta_1 G$, we find for the remaining *spherical aberration* a circle of radius 0″·0004 at distance ·000001 before the axial focus.

For faults that depend on obliquity, I shall take as standard

$$\beta = \cdot 01 = \tan^{-1} 34' \, 22''\cdot 6,$$

but shall also give the results for $\beta = \tan 30'$, and $= \tan 60'$. We have then for the *radius of the comatic circle*

$\beta = \tan 30'$.	$\beta = \tan 34'\cdot 4$.	$\beta = \tan 60'$.
$-0''\cdot 0042$	$-0''\cdot 0048$	$-0''\cdot 0083$.

For the *radius of the focal circle*

$\beta = \tan 30'$.	$\beta = \tan 34'\cdot 4$.	$\beta = \tan 60'$.
$+0''\cdot 282$	$+0''\cdot 370$	$+1''\cdot 127$.

For the *radius of the curvature of the field*, $-162\cdot 817$; and hence for the displacement of the focal circle from the plane through the axial principal focus

$\beta = \tan 30'$.	$\beta = \tan 34'\cdot 4$.	$\beta = \tan 60'$.
$-\cdot 000006$	$-\cdot 000008$	$-\cdot 000024$.

Finally, for the distortional displacement

$\beta = \tan 30'$.	$\beta = \tan 34'\cdot 4$.	$\beta = \tan 60'$.
$+4''\cdot 48$	$+6''\cdot 75$	$+35''\cdot 89$.

It will be recalled that the linear unit is supposed to be 100 inches.*

We conclude that spherical aberration, coma, and curvature of the field are now completely insensible, and that stars would be represented by strictly circular images of diameter 0·56 seconds at a distance of 30 minutes from the centre of the field, and 2·25 seconds at 1 degree distance. No images at present obtained with any telescope, at the middle of the field, where all obliquity-faults are absent, are sensibly less than 1 second in diameter. Hence this also is completely satisfactory up to a diameter of field of $1\frac{1}{2}$ degrees, or even more. There remains distortion, which requires examination. This can be calculated precisely and applied as a correction to measures made, along with differential refraction and other unavoidable corrections. Hence, even if its amount is very considerable it can be dealt with in a way that will not vitiate the use of the telescope. It is possible, indeed, that a correction for distortion requires to be applied to other telescopes now in use, especially those in which the lenses of the object-glass are separated. It is instructive to look into the contributions of the different surfaces to the total of δ_3H. The most remarkable is $-132\cdot 4$ units from the surface (6) which is nearly a plane surface. This is an obliquity-constituent, and would be present if the surface were a perfect plane. We see by examining the normal scheme next preceding the surface (6) that the original obliquity, β, of the ray is increased nearly

[* *Note added March 8, 1913.*—It is of interest to add that these conclusions have been checked by trigonometrical calculations also, made by Mr. A. E. CONRADY at the instance of one of the Referees.]

seven fold before impact upon this surface. It is this that produces the large coefficient.

It might be possible, with these numbers before one, to rearrange the general plan of the surfaces so as to produce a smaller value of $\delta_3 H$, but as explained above, it is not essential to do so in a telescope which is not likely to be used for exact measures over a field of more than 30 minutes radius.

We now return to the question of achromatism. We shall first verify that as far as the normal scheme goes, the achromatism which was secured for the scheme of p. 54, has not been sensibly impaired by the changes since made in the distribution of curvatures between the surfaces. Writing down only the surfaces, we have

$$\mu = 1{\cdot}5352.$$

Normal Schemes for $n+\delta n$.

Surface O_0	{+1·000000,	* ;	−·500000,	+1·000000},
Surfaces O_0, O_2	{+ ·339933,	+1·320133 ;	−·381696,	+ ·433886},
,, O_0, O_2, O'_2	{+ ·332299,	+1·328811 ;	−·081365,	+1·634860},
,, $O_0 \ldots O''_2$	{+ ·330671,	+1·361508 ;	−·208547,	+2·165477},
,, $O_0 \ldots O_4$	{+ ·141569,	+ ·3325076 ;	−·101641,	+2·213876},
,, $O_0 \ldots O'_4$	{+ ·140298,	+3·352749 ;	−·002245,	+7·074002},
,, $O_0 \ldots O_6$	{+ ·140287,	+3·388119 ;	−·000281,	+4·636412},
,, $O_0 \ldots O'_6$	{+ ·140283,	+3·457665 ;	−·196540,	+2·284156},
$O_0 \ldots F'_7$	{ * ,	+5·088015 ;	−·196540,	+2·284156}.

. . . . (37)

By comparing this with the schemes (35), p. 58, it will be seen that the rays of different refractive index separate decidedly in the course of their passage through the instrument before they are brought together at their common principal focus. The final agreement was to be expected as it was within our control, as far as the normal schemes were concerned, but it now remains to be considered whether there is any sensible chromatic difference of aberrations; this is found by recalculating the aberration coefficients with refractive index 1·5352 in place of 1·5200. The results are as follows:—

$n+\delta n$.

$$\delta_1 G = -{\cdot}00018, \quad \delta_2 G = +{\cdot}00846, \quad \delta_3 G = -{\cdot}45888.$$

$$\delta_1 H = +{\cdot}00844, \quad \delta_2 H = +1{\cdot}39705, \quad \delta_3 H = +351{\cdot}826. \quad \ldots \quad (38)$$

Interpreting these, as on p. 59, we conclude that for

$$\mu = 1{\cdot}5352.$$

Radius of Least Circle of Aberration, $-0''{\cdot}007$.

	$\beta = \tan 30'$.	$\beta = \tan 34'{\cdot}4$.	$\beta = \tan 60'$.
Comatic radius	$+0''{\cdot}0599$	$+0''{\cdot}0686$	$+\ 0''{\cdot}1197$
Focal radius	$+0''{\cdot}431$	$+0''{\cdot}566$	$+\ 1''{\cdot}724$
Distortion	$+4''{\cdot}74$	$+7''{\cdot}13$	$+37''{\cdot}91$
Displacement of focal circle .	$-\ {\cdot}000159$	$-\ {\cdot}000240$	$-\ {\cdot}000638$.

Radius of curvature of field, $-5{\cdot}423$.

The effect of the distortion at $\beta = {\cdot}01$ will be to draw out the image into a small spectrum of length $7''{\cdot}13 - 6''{\cdot}75 = 0''{\cdot}38$. The radius of curvature of the field is decidedly changed; but the effect of the change as shown in the corresponding displacement of the image-circle is not considerable.

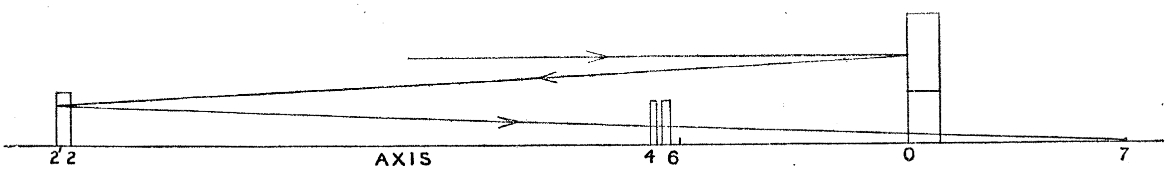

Fig. 2. Whole instrument. Scale 1 : 30.

It will be remarked that all these numbers run in the sense of increasing the aberrations; as there is no minimum property about the original index 1·52, we conclude that the aberrations for smaller indices would be proportionately diminished, and we see that it would have been better to have secured exact agreement for the larger index in place of the smaller one. In estimating the effect we may, for instance, take the following values, which are the indices for CHANCE's *hard crown* glass:—

Ray	C,	D,	F,	G,
Index	1·5150,	1·5175,	1·5235,	1·5284;

that is to say, with such a glass two-thirds of the excesses shown in the table above, over the results of p. 60, would cover all chromatic differences. There appears to be nothing in any of them that calls for a revision of the calculations.

Now let us turn to the question of the actual sizes and places of the mirrors and lenses in respect to the passage of a ray through the instrument. Calculate from the normal schemes, p. 58, for $b = +\cdot20$, and $\beta = -\cdot01, 0, +\cdot01$ respectively, the value of b' at each surface and also at the focal plane F'_7; this will give the necessary apertures for complete inclusion of all rays from the great mirror, up to these limits of obliquity. We find as follows:—

Value of Semi-aperture.

Surface.	$\beta = -\cdot01$.	$\beta = \cdot00$.	$\beta = +\cdot01$.
0	$+\cdot200$	$+\cdot200$	$+\cdot200$,
2	$+\cdot055$	$+\cdot068$	$+\cdot081$,
2′	$+\cdot054$	$+\cdot067$	$+\cdot080$,
2″	$+\cdot053$	$+\cdot066$	$+\cdot079$,
4	$-\cdot005$	$+\cdot028$	$+\cdot061$,
4′	$-\cdot006$	$+\cdot028$	$+\cdot062$,
6	$-\cdot006$	$+\cdot028$	$+\cdot062$,
6′	$-\cdot007$	$+\cdot028$	$+\cdot063$,
7′	$-\cdot051$	0	$+\cdot051$. . . . . (39)

Hence if the great mirror is 40 inches in diameter, the reverser requires to be 16·2 inches, the first face of the corrector 12·2 inches, and the last face 12·6 inches; the diameter of the image at the focal plane would be 10·2 inches.

It is necessary to verify that the corrector does not cut out any rays coming from the great mirror to the reverser. By the data on p. 54, the first face of the corrector is at a distance $+\cdot413750$ beyond the surface of the great mirror. Calculating the value of y' along the ray $y' = \beta'x' + b'$, for this value of x', where b', β' are taken from the normal scheme for the ray between the surfaces O_0 and O_2 we have

Value of b.	$\beta = -\cdot01$.	$\beta = \cdot00$.	$\beta = +\cdot01$.
$+\cdot200$	$+\cdot154$	$+\cdot159$	$+\cdot163$,
$+\cdot081$	$+\cdot061$	$+\cdot064$	$+\cdot068$. . . . (40)

Thus the ray which just cleared the reverser on its way to the great mirror would clear the corrector on its return.

Allowing that ·085 of the radius of the great mirror is unavailable the effective aperture-ratio is reduced from $40/508\cdot8 = 1 : 12\cdot72$ to $36\cdot28/508\cdot8 = 1 : 14\cdot05$.

The following table shows the inclinations of the ray to the axis of the telescope between the various surfaces:—

Inclination of Extreme Ray to Axis.

For $b = +\cdot 200$.			$\beta = -\cdot 01$.	$\beta = \cdot 00$.	$\beta = +\cdot 01$.
			°	°	°
Before surface O_0			$-0\cdot 6$	$0\cdot 0$	$+0\cdot 6$,
Between O_0	and	O_2	$-6\cdot 3$	$-5\cdot 7$	$-5\cdot 1$,
,, O_2	,,	O'_2	$-4\cdot 6$	$-4\cdot 4$	$-4\cdot 1$,
,, O'_2	,,	O''_2	$-1\cdot 9$	$-1\cdot 0$	$0\cdot 0$,
,, O''_2	,,	O_4	$-3\cdot 6$	$-2\cdot 4$	$-1\cdot 1$,
,, O_4	,,	O'_4	$-2\cdot 5$	$-1\cdot 2$	$-0\cdot 1$,
,, O'_4	,,	O_6	$-4\cdot 1$	$-0\cdot 1$	$+3\cdot 9$,
,, O_6	,,	O'_6	$-2\cdot 7$	$0\cdot 0$	$+2\cdot 6$,
,, O'_6	,,	F'_7	$-3\cdot 6$	$-2\cdot 3$	$-0\cdot 9$. . (41)

The inclinations of the ray to the normals of the surface ($_{2n}$) are given by $\beta_{2n}+b_{2n}B_{2n}$, $\beta'_{2n}+b_{2n}B_{2n}$ which may be calculated at once from the normal schemes; but note that as these include reversals for the case of a mirror we must then take in place of the latter $\beta'_{2n}-b_{2n}B_{2n}$:—

Inclination of Extreme Ray to Normal to Surface.

For $b = +\cdot 200$.	$\beta = -\cdot 01$.	$\beta = \cdot 00$.	$\beta = +\cdot 01$.
	°	°	°
O_0	$\begin{cases}-3\cdot 4\\-3\cdot 4\end{cases}$	$-2\cdot 9$ $-2\cdot 9$	$\left.\begin{matrix}-\ 2\cdot 3\\-\ 2\cdot 3\end{matrix}\right\}$
O_2	$\begin{cases}-4\cdot 8\\-3\cdot 2\end{cases}$	$-3\cdot 9$ $-2\cdot 6$	$\left.\begin{matrix}-\ 3\cdot 0\\-\ 1\cdot 9\end{matrix}\right\}$
O'_2	$\begin{cases}-3\cdot 3\\-3\cdot 3\end{cases}$	$-2\cdot 7$ $-2\cdot 7$	$\left.\begin{matrix}-\ 2\cdot 1\\-\ 2\cdot 1\end{matrix}\right\}$
O''_2	$\begin{cases}-3\cdot 3\\-5\cdot 0\end{cases}$	$-2\cdot 8$ $-4\cdot 2$	$\left.\begin{matrix}-\ 2\cdot 3\\-\ 3\cdot 3\end{matrix}\right\}$
O_4	$\begin{cases}-3\cdot 4\\-2\cdot 3\end{cases}$	$-3\cdot 5$ $-2\cdot 3$	$\left.\begin{matrix}-\ 3\cdot 6\\-\ 2\cdot 4\end{matrix}\right\}$
O'_4	$\begin{cases}-3\cdot 1\\-4\cdot 7\end{cases}$	$+2\cdot 1$ $+3\cdot 2$	$\left.\begin{matrix}+\ 7\cdot 3\\+11\cdot 0\end{matrix}\right\}$
O_6	$\begin{cases}-4\cdot 1\\-2\cdot 7\end{cases}$	$-0\cdot 1$ $-0\cdot 1$	$\left.\begin{matrix}+\ 3\cdot 8\\+\ 2\cdot 5\end{matrix}\right\}$
O'_6	$\begin{cases}-1\cdot 7\\-2\cdot 6\end{cases}$	$-4\cdot 2$ $-6\cdot 4$	$\left.\begin{matrix}-\ 6\cdot 8\\-10\cdot 2\end{matrix}\right\}$ . . (42)

Thus the greatest angle of incidence is 11·0 degrees upon the second surface of the first lens of the corrector. This is much below what is permitted in the construction of the object glass of a refractor; we find, for example, in STEINHEIL and VOIT's 'Handbuch,' with an aperture ratio of 1 : 12, the angle of incidence of extreme rays, originally *parallel to the axes*, upon the first surface if the flint-lens exceeds 15 degrees.

I would add a few remarks upon the problems presented by the construction of such a telescope, or at any rate, of its optical parts. It requires the production of a great mirror and three lenses which shall be in due relation to one another. None of the sizes or curves go outside what has already been made; and whenever a refractor is made, three of the surfaces must be turned out in agreement with the fourth. Hence there is no new difficulty in making and the problem is essentially a question of testing. The testing must be optical and not mechanical, for the former far outruns the latter in delicacy—it is said ten times. And because there are so many surfaces it would be essential to test them independently of one another. In the lenses, four out of the six surfaces are concave and spherical and can be tested

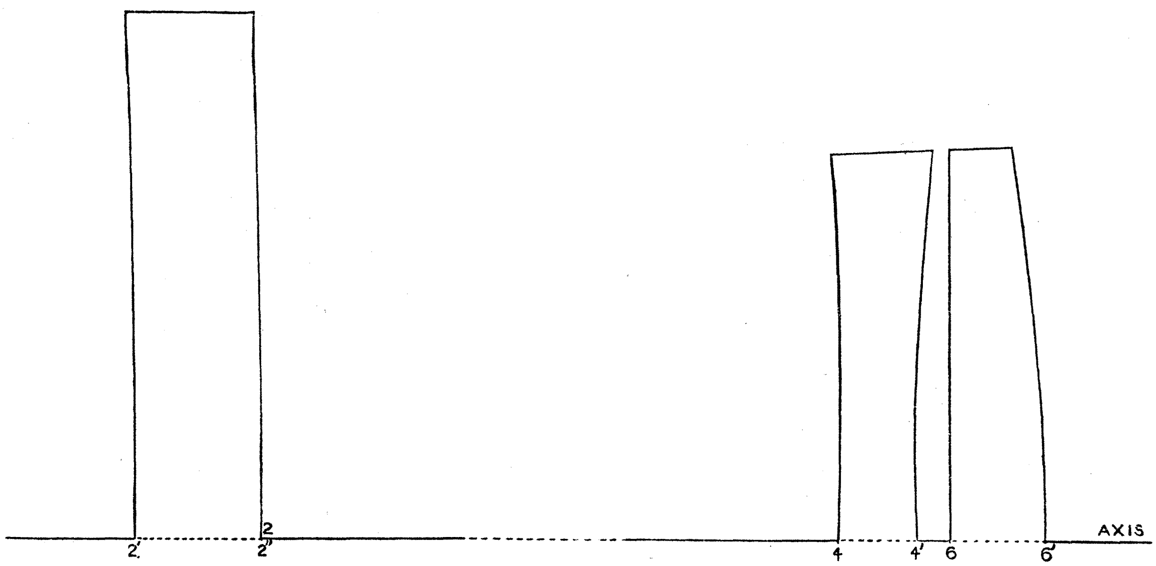

Fig. 3. Reverser and corrector. Scale 1 : 3.

with reflected light. The great mirror is neither a sphere nor a paraboloid, but its radius of curvature for different zones can be laid down, and each zone tested for agreement with this, just as in making a paraboloid. There remain then two convex surfaces, and the question of figuring the lens-surfaces to allow for inequalities of refractive index within the glass. These are matters for the skill of the maker and it would seem a not unreasonably difficult task.

I add a plan of the whole instrument and, upon a larger scale, of the reverser and corrector, and also the final specification, collected from pp. 60, 62, but making the unit 1 inch. For comparison the field of a Newtonian of the same aperture and focal length is added. It may be recalled that the displacement of the centre of the comatic circle is twice the comatic radius. For an uncorrected Cassegrain the field would be very much the same as for a Newtonian of the same aperture but of focal length equal to that of the great mirror, except in respect to curvature and distortion, see p. 41.

I would express my acknowledgments to Mr. R. W. Wrigley who helped me to perform many of the calculations.

Final Scheme.

Great mirror—

Aperture	$2a_0 = 40,$
Radius of curvature	$R_0 = -400{\cdot}000,$
	$\epsilon_0 = +{\cdot}16468.$
	$d_1 = +132{\cdot}013.$

Reverser—

Aperture	$2a_2 = 16{\cdot}2,$
First surface	$R_2 = +211{\cdot}603,$
Silvered surface	$R'_2 = +221{\cdot}289,$
Thickness	$t_2 = 2{\cdot}000.$
	$d_3 = +90{\cdot}676.$

Corrector, 1st lens—

Aperture	$2a_4 = 12{\cdot}2,$
First surface	$R_4 = -144{\cdot}298,$
Second surface	$R'_4 = +48{\cdot}824,$
Thickness	$t_4 = 1{\cdot}250.$
	$d_5 = +0{\cdot}500.$

Corrector, 2nd lens—

Aperture	$2a_6 = 12{\cdot}6,$
First surface	$R_6 = -4138{\cdot}559,$
Second surface	$R'_6 = -38{\cdot}285,$
Thickness	$t_6 = 1{\cdot}500.$
	$d_7 = +71{\cdot}377.$

Focal length	$f'_7 = +508{\cdot}802.$
Distance of principal focus beyond surface of great mirror	$+33{\cdot}290.$
Whole length of instrument	$167{\cdot}3.$

Specification of Field at $\beta = {\cdot}01 = \tan 34'{\cdot}4$.

	$\mu = 1{\cdot}5200.$	$\mu = 1{\cdot}5352.$	[Newtonian.]
	$''$	$''$	$''$
Radius of least circle of aberration . .	0·000	−0·007	0·00
Radius of comatic circle	−0·005	+0·069	+0·80
Radius of focal circle	+0·370	+0·566	−0·41
Distortional displacement	+6·75	+7·13	0·00.
Curvature of field	−1/16282	−1/542·3	−1/508·8.